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WELL-POSEDNESS OF HIGHER-ORDER CAMASSA–HOLM EQUATIONS

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Abstract. We consider higher-order Camassa–Holm equations describing exponential curves of the manifold of smooth orientation preserving diffeomorphisms of the unit circle in the plane. We establish the existence of a strongly continuous semigroup of global weak solutions. We also present some invariant spaces under the action of that semigroup. Moreover, we prove a "weak equals strong" uniqueness result.

1. Introduction

Consider the unit circle S^1 in the plane and the manifold $\mathcal{D}$ of the smooth orientation-preserving diffeomorphisms of S^1. Following [14] we study the equation for the exponential curves on $\mathcal{D}$ using the Riemannian structure induced by the Sobolev inner product $(\,\cdot\,,\,\cdot\,)_{H^k(\mathbb{R})}$, $k \in \mathbb{N}$ (where we identify $H^0(\mathbb{R})$ and $L^2(\mathbb{R})$). Let $k \in \mathbb{N}$ and

$$\Gamma : t \geq 0 \mapsto u(t,\,\cdot\,) \in \mathcal{D}$$

be a curve. It is a an exponential curve if it satisfies the following equation [14, (3.7)]

$$\partial_t u = B_k(u,u), \qquad t>0,\ x\in\mathbb{R}, \tag{1.1}$$

where (see [14, (3.2), (3.3), and Proof of Theorem 2])

$$\begin{aligned} B_k(u,u) &:= A_k^{-1} C_k(u) - u\partial_x u, \\ A_k(u) &:= \sum_{j=0}^{k} (-1)^j \partial_x^{2j} u, \\ C_k(u) &:= -uA_k\big(\partial_x u\big) + A_k\big(u\partial_x u\big) - 2\partial_x u A_k(u). \end{aligned}$$

In the cases $k=0$ and $k=1$, (1.1) becomes the inviscid Burgers equation [24]

$$\partial_t u + 3u\partial_x u = 0, \tag{1.2}$$

and the Camassa–Holm equation [2, 9]

$$\partial_t u - \partial_t \partial_x^2 u + 3u\partial_x u = 2\partial_x u \partial_x^2 u + u\partial_x^3 u, \tag{1.3}$$

respectively (see [14, Examples 1 and 2]). This infinite sequence of higher-order Camassa–Holm equations is distinct from what is normally called the Camassa–Holm hierarchy, where the equations beyond the Camassa–Holm equation itself are non-local and all equations are completely integrable in the sense that one can find

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a zero-curvature formulation for each equation in the hierarchy. Indeed, that is the main mechanism behind their construction. For details about the Camassa–Holm hierarchy, see [20, 21] and references therein.

In this paper we study the wellposedness of the equation (1.1). In particular, we show that it possesses a globally defined weak solution u in $C([0,\infty);C^{k-1}(\mathbb{R}))\cap L^\infty\big([0,\infty);H^k(\mathbb{R})\big)$ when the initial data $u_0\in H^k(\mathbb{R})$, $\partial_x^k u_0\in L^p(\mathbb{R})$, for some $2<p<\infty$, see Definition 2.3 and Theorem 2.4. Furthermore, we show the existence of a semigroup S_t associated with the problem in the sense that $u=S_t(u_0)$ solves the equation (1.1) with initial data u_0. The semigroup is continuous in the following sense: If $u_{0,n}\to u_0$ in $H^k(\mathbb{R})$, then $S(u_{0,n})\to S(u_0)$ in $L^\infty([0,T];H^k(\mathbb{R}))$, see Theorem 2.4.

Similar results holds for the Camassa–Holm equation (1.3) in the case $k=1$ [5]. This equation models the propagation of unidirectional shallow water waves on a flat bottom, and $u(t,x)$ represents the fluid velocity at time t in the horizontal direction x [2, 23]. The Camassa–Holm equation possesses a bi-Hamiltonian structure (and thus an infinite number of conservation laws) [19, 2] and is completely integrable [2, 1, 12, 8]. Moreover, it has an infinite number of solitary wave solutions, called *peakons* due to the discontinuity of their first derivatives at the wave peak, interacting like solitons: $u(t,x)=ce^{-|x-ct|}$, $c\in\mathbb{R}$. From a mathematical point of view the Camassa–Holm equation is well studied. Local well-posedness results are proved in [10, 22, 25, 27]. It is also known that there exist global solutions for a particular class of initial data and also solutions that blow up in finite time for a large class of initial data [7, 10, 9]. Here blow up means that the slope of the solution becomes unbounded while the solution itself stays bounded. More relevant for the present paper, we recall that existence and uniqueness results for global weak solutions of (1.3) are proved in [11, 13, 29, 30, 15, 16], see also [5].

On the other hand we recall that the solutions of the Burgers equation (1.2) in the case $k=0$ experience shock formation and indeed it is well-posed in the space $L^\infty([0,\infty);BV(\mathbb{R}))$. Let us mention the Degasperis–Procesi equation [17, 18]

$$\partial_t u-\partial_t\partial_x^2u+4u\partial_x u=3\partial_x u\partial_x^2 u+u\partial_x^3 u. \tag{1.4}$$

It appears to be similar to the Camassa–Holm equation (1.3), but its solutions are in general discontinuous, see [6] and the references cited therein.

To keep the presentation short, details are presented for the case $k=2$ only. In Appendices A and B we show how to extend the theory to general $k>2$.

The paper is organized as follows. In Section 2 we introduce the equations and state the main result. The existence result is obtained as a singular limit of a viscous regularization. The necessary a priori estimates are treated in Section 3. In Section 4 stability with respect to the viscous regularization is proved using a homotopy argument. The necessary compactness arguments as well as regularity of the solution is obtained in Section 6. In Section 7 we prove a "weak equals strong" uniqueness result. Appendices A and B deal with the general case $k>2$.

2. The governing equations and the main theorem

We construct a family of higher-order Camassa–Holm equations as follows. Let $k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. Consider the equation

$$\partial_t u=B_k(u,u) \tag{2.1}$$

where $u=u(t,x)\colon[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is the unknown function and

$$B_k(u,u):=A_k^{-1}C_k(u)-u\partial_x u,$$

$$A_k(u) := \sum_{j=0}^{k} (-1)^j \partial_x^{2j} u, \tag{2.2}$$
$$C_k(u) := -uA_k\big(\partial_x u\big) + A_k\big(u\partial_x u\big) - 2\partial_x u A_k(u).$$

It turns out that the operator $C_k(u)$ is a total derivative, that is, there exists a differential polynomial in u denoted by $\mathcal{F}_k$ such that

$$C_k(u) = -\partial_x \mathcal{F}_k(u).$$

One can see this as follows

$$\begin{aligned}
-\mathcal{F}_k(u) &= \int_{-\infty}^{x} C_k(u(\xi))\, d\xi \\
&= \int_{-\infty}^{x} \big(- uA_k(\partial_x u) + A_k(u\partial_x u) - 2\partial_x u A_k(u)\big) d\xi \\
&= \sum_{j=0}^{k} (-1)^j \int_{-\infty}^{x} \big(- u\partial_x^{2j}\partial_x u + \partial_x^{2j}(u\partial_x u) - 2\partial_x u \partial_x^{2j} u\big) d\xi \\
&= \frac{1}{2}\sum_{j=0}^{k} (-1)^j \partial_x^{2j}(u^2) - \sum_{j=0}^{k} (-1)^j \int_{-\infty}^{x} \big(u\partial_x^{2j}\partial_x u + 2\partial_x u \partial_x^{2j} u\big) d\xi.
\end{aligned}$$

Lemma A.2 shows that indeed the integrand in each term is a total derivative, making $\mathcal{F}_k(u)$ a differential polynomial in u.

Remark 2.1. *The operator A_k^{-1} has a convolution structure, more precisely*

$$A_k^{-1}(f)(x) = \int_{\mathbb{R}} G_k(x-y) f(y)\, dy, \qquad x \in \mathbb{R}, \tag{2.3}$$

where G_k has Fourier transform $\widehat{G}_k$ given by

$$\widehat{G}_k(\zeta) = \frac{1}{1+\zeta^2+\cdots+\zeta^{2k}}, \qquad \zeta \in \mathbb{R}.$$

We also have

$$G_k \geq 0, \qquad \|G_k\|_{W^{2k-1,1}(\mathbb{R})}, \|G_k\|_{W^{2k-1,\infty}(\mathbb{R})} \leq C_0, \tag{2.4}$$

for some constant $C_0 > 0$. In the special case $k = 1$ we find

$$G_1(x) = \frac{1}{2} e^{-|x|}.$$

We will repeatedly use that

$$\partial_x^j A_k(u) = A_k(\partial_x^j u)$$

as well as

$$\int_{\mathbb{R}} vA_k(w)\, dx = \int_{\mathbb{R}} A_k(v) w\, dx.$$

Example 2.2. (2.1) *reads in the cases $k = 0, 1, 2, 3$ as follows* [14].
(i) For $k = 0$ we find the following:

$$\partial_t u + u\partial_x u = -\partial_x(u^2) \quad \text{or} \quad \partial_t u + 3u\partial_x u = 0,$$

which constitutes the inviscid Burgers equation, and

$$A_0(u) = u, \quad C_0(u) = -2u\partial_x u, \quad \mathcal{F}_0(u) = u^2.$$

(ii) For $k = 1$ we obtain the following equation:

$$\partial_t u + u\partial_x u = -\partial_x A_1^{-1}\big(u^2 + \frac{1}{2}(\partial_x u^2)\big)$$

or

$$\partial_t u - \partial_t \partial_x^2 u + 3u\partial_x u = 2\partial_x u \partial_x^2 u + u\partial_x^3 u,$$

which is the Camassa–Holm equation. Furthermore,

$$A_1(u) = u - \partial_x^2 u, \quad C_1(u) = -2u\partial_x u - \partial_x u \partial_x^2 u, \quad \mathcal{F}_1(u) = u^2 + \frac{1}{2}(\partial_x u)^2.$$

(iii) For $k = 2$, equation (2.1) becomes

$$\partial_t u + u\partial_x u = -A_2^{-1}\partial_x \left[u^2 + \frac{1}{2}\big(\partial_x u\big)^2 - \frac{1}{2}\big(\partial_x^2 u\big)^2 - 3\partial_x\big(\partial_x u \partial_x^2 u\big)\right], \tag{2.5}$$

or equivalently

$$\partial_t u - \partial_t \partial_x^2 u + \partial_t \partial_x^4 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u + 2\partial_x u \partial_x^4 u + u\partial_x^5 u = 0. \tag{2.6}$$

In particular,

$$\begin{aligned}
A_2(u) &= \partial_x^4 u - \partial_x^2 u + u,\\
C_2(u) &= -uA_2\big(\partial_x u\big) + A_2\big(u\partial_x u\big) - 2\partial_x u A_2(u)\\
&= -\partial_x u \partial_x^2 u + 10\partial_x^2 u \partial_x^3 u + 3\partial_x u \partial_x^4 u - 2u\partial_x u,\\
\mathcal{F}_2(u) &= -\int_{-\infty}^{x} C_2(u)\, dx\\
&= u^2 + \frac{1}{2}\big(\partial_x u\big)^2 - \frac{7}{2}\big(\partial_x^2 u\big)^2 - 3\partial_x u \partial_x^3 u\\
&= u^2 + \frac{1}{2}\big(\partial_x u\big)^2 - \frac{1}{2}\big(\partial_x^2 u\big)^2 - 3\partial_x\big(\partial_x u \partial_x^2 u\big).
\end{aligned}$$

(iv) For $k = 3$, equation (2.1) becomes

$$\begin{aligned}
\partial_t u + u\partial_x u = -A_2^{-1}\partial_x \bigg[u^2 &+ \frac{1}{2}\big(\partial_x u\big)^2 - \frac{7}{2}\left(\partial_x^2 u\right)^2 - 3\partial_x u \partial_x^3 u\\
&+5\partial_x u \partial_x^5 u + 16\partial_x^2 u \partial_x^4 u + \frac{19}{2}\left(\partial_x^3 u\right)^2\bigg],
\end{aligned} \tag{2.7}$$

or equivalently

$$\begin{aligned}
\partial_t u - \partial_t \partial_x^2 u + \partial_t \partial_x^4 u - \partial_t \partial_x^6 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u&\\
+ 2\partial_x u \partial_x^4 u + u\partial_x^5 u - 2\partial_x u \partial_x^6 u - u\partial_x^7 u &= 0.
\end{aligned}$$

In particular,

$$\begin{aligned}
A_3(u) &= -\partial_x^6 u + \partial_x^4 u - \partial_x^2 u + u = A_2(u) - \partial_x^6 u,\\
C_3(u) &= -uA_3\big(\partial_x u\big) + A_3\left(u\partial_x u\right) - 2\partial_x u A_3(u)\\
&= C_2(u) - 35\partial_x^3 u \partial_x^4 u - 21\partial_x^2 u \partial_x^5 u - 5\partial_x u \partial_x^6 u,\\
\mathcal{F}_3(u) &= -\int_{-\infty}^{x} C_3(u)\, dx\\
&= u^2 + \frac{1}{2}\big(\partial_x u\big)^2 - \frac{7}{2}\left(\partial_x^2 u\right)^2 - 3\partial_x u \partial_x^3 u\\
&\quad + 5\partial_x u \partial_x^5 u + 16\partial_x^2 u \partial_x^4 u + \frac{19}{2}\left(\partial_x^3 u\right)^2.
\end{aligned}$$

We are interested in the Cauchy problem

$$\begin{cases}\partial_t u = B_k(u, u), & (t, x) \in (0, \infty) \times \mathbb{R},\\ u(0, x) = u_0(x), & x \in \mathbb{R}\end{cases} \tag{2.8}$$

in the case $k \geq 2$. We will assume

$$u_0 \in H^k(\mathbb{R}), \quad \partial_x^k u_0 \in L^p(\mathbb{R}) \quad \text{for some } 2 < p < \infty. \tag{2.9}$$

For the definition of weak solutions of (2.8) we reformulate the equation as a system of an hyperbolic equation and an higher order elliptic one, namely

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x P = 0, \\ A_k(P) = \mathcal{F}_k(u). \end{cases} \tag{2.10}$$

This formulation is formally equivalent to (2.8).

Definition 2.3. *We call a function $u\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ a weak solution of* (2.8) *if*

(*i*) $u \in C([0,\infty); C^{k-1}(\mathbb{R})) \cap L^\infty\big([0,\infty); H^k(\mathbb{R})\big)$*;*
(*ii*) *u satisfies* (2.10) *in the sense of distributions;*
(*iii*) $u(0,x) = u_0(x)$ *for every $x\in\mathbb{R}$;*
(*iv*) $\|u(t,\cdot)\|_{H^k(\mathbb{R})} \le \|u_0\|_{H^k(\mathbb{R})}$, *for each $t>0$.*

Our main result is the following.

Theorem 2.4. *Let $2<p<\infty$. There exists a strongly continuous semigroup of solutions*

$$S\colon [0,\infty)\times\mathcal{H}_{k,p} \to C([0,\infty); C^{k-1}(\mathbb{R})) \cap L^\infty\big([0,\infty); H^k(\mathbb{R})\big)$$

associated with the Cauchy problem (2.8)*, where*

$$\mathcal{H}_{k,p} := \Big\{ f\in H^k(\mathbb{R}) \mid \partial_x^k f \in L^p(\mathbb{R})\Big\}.$$

More precisely, we have

(*j*) *for each $u_0\in\mathcal{H}_{k,p}$ the map $u(t,x) = S_t(u_0)(x)$ is a weak solution of* (2.8) *according to Definition* 2.3*;*
(*jj*) *the semigroup is stable with respect to the initial condition:*

$$u_{0,n}\to u_0 \text{ in } H^k(\mathbb{R}) \text{ implies } S(u_{0,n})\to S(u_0) \text{ in } L^\infty([0,T];H^k(\mathbb{R})), \tag{2.11}$$

for every $\{u_{0,n}\}_{n\in\mathbb{N}}\subset\mathcal{H}_{k,p}$, $u_0\in\mathcal{H}_{k,p}$, $T>0$.

Moreover, the spaces $H^{k+1}(\mathbb{R})$ and $\mathcal{H}_{k,r}$, $2\le r<\infty$ are invariant under the action of S, i.e.,

$$S([0,T]\times H^{k+1}(\mathbb{R})) \subset L^\infty([0,T];H^{k+1}(\mathbb{R})), \tag{2.12}$$

$$S([0,T]\times\mathcal{H}_{k,r}) \subset L^\infty([0,T];\mathcal{H}_{k,r}), \quad 2\le r<\infty, \tag{2.13}$$

for each $T>0$.

Moreover we show the uniqueness of the solution of the Cauchy problem (2.8) within the class of the maps with bounded second spatial derivative. A similar result was proved in [30], in the case $k=1$, for the Camassa–Holm equation. More precisely, we prove the following “weak equals strong” uniqueness principle.

Theorem 2.5. *Assume $k=2$. Let u be a weak solution of the Cauchy problem* (2.8) *in the sense of Definition* 2.3*. If there exists a map $b\in L^1([0,T])$, $T>0$, such that*

$$\left\|\partial_x^2 u(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \le b(t), \qquad t\ge 0,$$

then, u is unique within the class of the maps satisfying such a condition.

In particular, here we assume $b\in L^1([0,T])$, $T>0$, and in [30] when $k=1$, the authors assumed $b\in L^2([0,T])$, $T>0$.

One should observe that the behavior of the Camassa–Holm equation ($k=1$) is quite different from the behaviour of (2.8). Indeed the equation for $q=\partial_x^2 u$, which is a relevant quantity for (2.8), is

$$\partial_t q + u\partial_x q + \widetilde{P} = 0,$$

where $\widetilde{P}$ is a given function that will be defined later on. On the other hand, if u solves the Camassa–Holm equation (1.3), then $q = \partial_x u$, which is the corresponding relevant quantity, satisfies (P is a another given function)

$$\partial_t q + u\partial_x q + \frac{1}{2}q^2 - u^2 + P = 0,$$

which now contains the nonlinear term q^2.

We apply the following singular perturbation approach. Let $\varepsilon > 0$, and consider the system

$$\begin{cases} \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_x^2 u_\varepsilon, \\ A_k(P_\varepsilon) = \mathcal{F}_k(u_\varepsilon), \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x). \end{cases} \tag{2.14}$$

We call the solution $u_\varepsilon = u_\varepsilon(t,x)$ of (2.14) a *viscous approximant* to the solution $u = u(t,x)$ of (2.8). Furthermore, we shall assume

$$u_{0,\varepsilon} \in H^{k+1}(\mathbb{R}), \qquad \|u_{0,\varepsilon}\|_{H^k(\mathbb{R})} \le \|u_0\|_{H^k(\mathbb{R})}, \qquad u_{0,\varepsilon} \to u_0 \text{ in } H^k(\mathbb{R}). \tag{2.15}$$

Example 2.6. *The equations* (2.10) *and* (2.14) *read in the special cases* $k = 0,1,2,3$ *as follows.*
(i) For $k = 0$ *we find the following:*

$$\partial_t u + u\partial_x u + \partial_x P = 0, \quad P = u^2,$$

and

$$\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_x^2 u_\varepsilon, \quad P_\varepsilon = u_\varepsilon^2.$$

(ii) For $k = 1$ *we obtain the following equations*

$$\partial_t u + u\partial_x u + \partial_x P = 0, \quad P - \partial_x^2 P = u^2 + \frac{1}{2}(\partial_x u)^2,$$

and

$$\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_x^2 u_\varepsilon, \quad P_\varepsilon - \partial_x^2 P_\varepsilon = u_\varepsilon^2 + \frac{1}{2}(\partial_x u_\varepsilon)^2.$$

(iii) For $k = 2$ *we find*

$$\begin{aligned} &\partial_t u + u\partial_x u + \partial_x P = 0, \\ &\partial_x^4 P - \partial_x^2 P + P = u^2 + \frac{1}{2}\big(\partial_x u\big)^2 - \frac{1}{2}\big(\partial_x^2 u\big)^2 - 3\partial_x\big(\partial_x u \partial_x^2 u\big), \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_x^2 u_\varepsilon, \\ &\partial_x^4 P_\varepsilon - \partial_x^2 P_\varepsilon + P_\varepsilon = u_\varepsilon^2 + \frac{1}{2}\big(\partial_x u_\varepsilon\big)^2 - \frac{7}{2}\big(\partial_x^2 u_\varepsilon\big)^2 - 3\partial_x u_\varepsilon \partial_x^3 u_\varepsilon, \end{aligned} \tag{2.17}$$

or equivalently

$$\begin{aligned} &\partial_t u_\varepsilon - \partial_t \partial_x^2 u_\varepsilon + \partial_t \partial_x^4 u_\varepsilon + 3u_\varepsilon \partial_x u_\varepsilon - 2\partial_x u_\varepsilon \partial_x^2 u_\varepsilon \\ &\quad - u\partial_x^3 u_\varepsilon + 2\partial_x u_\varepsilon \partial_x^4 u_\varepsilon + u\partial_x^5 u_\varepsilon = \varepsilon \partial_x^2 u_\varepsilon - \varepsilon \partial_x^4 u_\varepsilon + \varepsilon \partial_x^6 u_\varepsilon. \end{aligned} \tag{2.18}$$

(iv) For $k = 3$ *we find*

$$\begin{aligned} &\partial_t u + u\partial_x u + \partial_x P = 0, \\ -\partial_x^6 P + \partial_x^4 P - \partial_x^2 P + P &= u^2 + \frac{1}{2}\big(\partial_x u\big)^2 - \frac{7}{2}\big(\partial_x^2 u\big)^2 - 3\partial_x u \partial_x^3 u \\ &\quad + 5\partial_x^2\big(\partial_x u \partial_x^3 u\big) + 6\partial_x\big(\partial_x^2 u \partial_x^3 u\big) - \frac{3}{2}\big(\partial_x^3 u\big)^2. \end{aligned} \tag{2.19}$$

Remark 2.7. *Introducing the quantity*

$$m := A_k(u), \qquad m_\varepsilon := A_k(u_\varepsilon),$$

we have, see [14], *that equations* (2.10) *and* (2.14) *equal*

$$\partial_t m + u\partial_x m + 2m\partial_x u = 0, \tag{2.20}$$

and

$$\partial_t m_\varepsilon + u_\varepsilon\partial_x m_\varepsilon + 2m_\varepsilon\partial_x u_\varepsilon = \varepsilon\partial_x^2 m_\varepsilon, \tag{2.21}$$

respectively.

3. Viscous approximants: Global existence and energy estimate

We begin with the existence of the viscous approximants to (2.8).

Lemma 3.1. *Assume* (2.9) *and* (2.15)*. Let* $\varepsilon > 0$*. Then there exists a unique global smooth solution* $u_\varepsilon = u_\varepsilon(t,x)$ *of the Cauchy problem* (2.14) *belonging to* $C([0,\infty); H^{k+1}(\mathbb{R}))$*.*

Proof. The proof of this statement is similar to the one of [4, Theorem 2.3], and is therefore omitted. □

Lemma 3.2 (Energy estimate)**.** *Assume* (2.9) *and* (2.15)*. The identity*

$$\|u_\varepsilon(t,\cdot)\|^2_{H^k(\mathbb{R})} + 2\varepsilon\int_0^t \left\|\partial_x u_\varepsilon(\tau,\cdot)\right\|^2_{H^k(\mathbb{R})} d\tau = \|u_{0,\varepsilon}\|^2_{H^k(\mathbb{R})} \tag{3.1}$$

holds for each $t \geq 0$ *and* $\varepsilon > 0$*. In addition,*

$$\|u_\varepsilon\|_{L^\infty([0,\infty)\times\mathbb{R})}, \dots, \left\|\partial_x^{k-1} u_\varepsilon\right\|_{L^\infty([0,\infty)\times\mathbb{R})} \leq \frac{1}{\sqrt{2}}\|u_0\|_{H^k(\mathbb{R})}, \tag{3.2}$$

for each $\varepsilon > 0$*.*

Proof. Fix $t > 0$. Multiplying the first equation of (2.14) by $A_k(u_\varepsilon)$ and integrating over $\mathbb{R}$, we get

$$\begin{aligned} &\int_{\mathbb{R}} \partial_t u_\varepsilon A_k(u_\varepsilon)dx - \varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon A_k(u_\varepsilon)dx \\ &\qquad = -\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon A_k(u_\varepsilon)dx - \int_{\mathbb{R}} \partial_x P_\varepsilon A_k(u_\varepsilon)dx. \end{aligned} \tag{3.3}$$

Integrating by parts we have for the left-hand side,

$$\begin{aligned} &\int_{\mathbb{R}} \partial_t u_\varepsilon A_k(u_\varepsilon)dx - \varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon A_k(u_\varepsilon)dx \\ &\qquad =\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^2_{H^k(\mathbb{R})} + \varepsilon\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{H^k(\mathbb{R})}, \end{aligned} \tag{3.4}$$

and, using the second equation of (2.14), we have for the right-hand side,

$$\begin{aligned} &-\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon A_k(u_\varepsilon)dx - \int_{\mathbb{R}} \partial_x P_\varepsilon A_k(u_\varepsilon)dx \\ &= -\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon A_k(u_\varepsilon)dx - \int_{\mathbb{R}} \partial_x(A_k(P_\varepsilon))u_\varepsilon dx \\ &= -\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon A_k(u_\varepsilon)dx + \int_{\mathbb{R}} C_k(u_\varepsilon)u_\varepsilon dx \\ &= -3\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon A_k(u_\varepsilon)dx - \int_{\mathbb{R}} u_\varepsilon^2 A_k\big(\partial_x u_\varepsilon\big) + \int_{\mathbb{R}} u_\varepsilon A_k\big(u_\varepsilon\partial_x u_\varepsilon\big)dx \\ &= -3\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon A_k(u_\varepsilon)dx + \int_{\mathbb{R}} \partial_x(u_\varepsilon^2)A_k(u_\varepsilon) + \int_{\mathbb{R}} A_k(u_\varepsilon)u_\varepsilon\partial_x u_\varepsilon dx = 0. \end{aligned} \tag{3.5}$$

Substituting (3.4) and (3.5) in (3.3),

$$\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^2_{H^k(\mathbb{R})} + 2\varepsilon\big\|\partial_x u_\varepsilon(t,\cdot)\big\|^2_{H^k(\mathbb{R})} = 0.$$

Integrating over $[0,t]$, we get (3.1). Finally, (3.2) is direct consequence of [26, Theorem 8.5], equations (2.15) and (3.1). □

4. Bounds on the source term P_ε and invariance properties with $k = 2$

From now on we assume $k = 2$. We show in Appendix B how to extend the proofs to the general case $k > 2$.

Using Remark 2.1, we may write

$$P_\varepsilon = P_{1,\varepsilon} + P_{2,\varepsilon}, \tag{4.1}$$

where

$$P_{1,\varepsilon}(t,x) := \int_{\mathbb{R}} G_2(x-y)\left[u_\varepsilon^2(t,y) + \frac{1}{2}\left(\partial_x u_\varepsilon(t,y)\right)^2 - \frac{1}{2}\left(\partial_x^2 u_\varepsilon(t,y)\right)^2\right]dy,$$

$$P_{2,\varepsilon}(t,x) := -3\int_{\mathbb{R}} G_2(x-y)\left[\left(\partial_x^2 u_\varepsilon(t,y)\right)^2 + \partial_x u_\varepsilon(t,y)\partial_x^3 u_\varepsilon(t,y)\right]dy.$$

Moreover, since G_2 is the Green's function of the operator A_2, we have

$$\begin{aligned}\partial_x^3 P_{2,\varepsilon}(t,x) &= -3\int_{\mathbb{R}} G_2'''(x-y)\left[\left(\partial_x^2 u_\varepsilon(t,y)\right)^2 + \partial_x u_\varepsilon(t,y)\partial_x^3 u_\varepsilon(t,y)\right]dy\\ &= -3\partial_x u_\varepsilon(t,x)\partial_x^2 u_\varepsilon(t,x)\\ &\quad - 3\int_{\mathbb{R}}\left(G_2''(x-y) - G_2(x-y)\right)\partial_x u_\varepsilon(t,y)\partial_x^2 u_\varepsilon(t,y)dy.\end{aligned}$$

Hence

$$\partial_x^3 P_\varepsilon = -3\partial_x u_\varepsilon \partial_x^2 u_\varepsilon + \partial_x^3 P_{1,\varepsilon} + P_{3,\varepsilon}, \tag{4.2}$$

where

$$P_{3,\varepsilon}(t,x) := -3\int_{\mathbb{R}}\left(G_2''(x-y) - G_2(x-y)\right)\partial_x u_\varepsilon(t,y)\partial_x^2 u_\varepsilon(t,y)dy,$$

for each $\varepsilon > 0$, $t \ge 0$, $x \in \mathbb{R}$.

Lemma 4.1. *Assume $k = 2$,* (2.9) *and* (2.15)*. The following inequalities hold*

$$\|P_\varepsilon(t,\cdot)\|_{W^{2,1}(\mathbb{R})},\ \|P_\varepsilon(t,\cdot)\|_{W^{2,\infty}(\mathbb{R})} \le 4C_0\|u_0\|^2_{H^2(\mathbb{R})}, \tag{4.3}$$

$$\|P_{1,\varepsilon}(t,\cdot)\|_{W^{4,1}(\mathbb{R})},\ \|P_{1,\varepsilon}(t,\cdot)\|_{W^{4,\infty}(\mathbb{R})} \le (6C_0+1)\|u_0\|^2_{H^2(\mathbb{R})}, \tag{4.4}$$

$$\|P_{2,\varepsilon}(t,\cdot)\|_{W^{2,1}(\mathbb{R})},\ \|P_{2,\varepsilon}(t,\cdot)\|_{W^{2,\infty}(\mathbb{R})} \le 2C_0\|u_0\|^2_{H^2(\mathbb{R})}, \tag{4.5}$$

$$\|\partial_x^3 P_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le (7C_0+3)\|u_0\|^2_{H^2(\mathbb{R})}, \tag{4.6}$$

$$\|P_{3,\varepsilon}(t,\cdot)\|_{W^{1,1}(\mathbb{R})},\ \|P_{3,\varepsilon}(t,\cdot)\|_{W^{1,\infty}(\mathbb{R})} \le 12C_0\|u_0\|^2_{H^2(\mathbb{R})}, \tag{4.7}$$

for each $t \ge 0$ and $\varepsilon > 0$.

Proof. Fix $t > 0$. We begin by proving (4.4). Observing that,

$$\partial_x^i P_{1,\varepsilon}(t,x) = \int_{\mathbb{R}} \frac{d^i G_2}{dx^i}(x-y)\left[u_\varepsilon^2 + \frac{1}{2}\left(\partial_x u_\varepsilon(t,y)\right)^2 - \frac{1}{2}\left(\partial_x^2 u_\varepsilon(t,y)\right)^2\right]dy,$$

from (2.4) and (3.2),

$$\begin{aligned}\|\partial_x^i P_{1,\varepsilon}(t,\cdot)\|_{L^p(\mathbb{R})} &\le \left\|\frac{d^i G_2}{dx^i}\right\|_{L^p(\mathbb{R})}\int_{\mathbb{R}}\left[u_\varepsilon^2 + \frac{1}{2}\left(\partial_x u_\varepsilon\right)^2 + \frac{1}{2}\left(\partial_x^2 u_\varepsilon\right)^2\right]dy\\ &\le C_0\|u(t,\cdot)\|^2_{H^2(\mathbb{R})} \le C_0\|u_0\|^2_{H^2(\mathbb{R})},\end{aligned} \tag{4.8}$$

for each $p \in \{1, \infty\}$, $i \in \{0, 1, 2, 3\}$. Recalling that G_2 is the Green's function of the operator A_2 (see Remark 2.1), we find

$$\partial_x^4 P_{1,\varepsilon} = \partial_x^2 P_{1,\varepsilon} - P_{1,\varepsilon} + u_\varepsilon^2 + \frac{1}{2}(\partial_x u_\varepsilon)^2 - \frac{1}{2}(\partial_x^2 u_\varepsilon)^2, \tag{4.9}$$

hence, (4.4) is a direct consequence of (3.1), (4.8), and (4.9).

We continue by proving (4.5). Observing that,

$$\begin{aligned}\partial_x^j P_{2,\varepsilon}(t,x) &= -3\int_{\mathbb{R}} \frac{d^j G_2}{dx^j}(x-y)\left[(\partial_x^2 u_\varepsilon(t,y))^2 + \partial_x u_\varepsilon(t,y)\partial_x^3 u_\varepsilon(t,y)\right] dy\\ &= -3\int_{\mathbb{R}} \frac{d^{j+1} G_2}{dx^{j+1}}(x-y)\partial_x u_\varepsilon(t,y)\partial_x^2 u_\varepsilon(t,y)dy,\end{aligned}$$

we conclude, using the Hölder inequality, (2.4) and (3.2), that

$$\begin{aligned}\|\partial_x^j P_{2,\varepsilon}(t,\cdot)\|_{L^p(\mathbb{R})} &\le \left\|\frac{d^{j+1}G_2}{dx^{j+1}}\right\|_{L^p(\mathbb{R})}\int_{\mathbb{R}} |\partial_x u_\varepsilon \partial_x^2 u_\varepsilon| dy\\ &\le C_0\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\ &\le C_0\|u(t,\cdot)\|^2_{H^2(\mathbb{R})} \le C_0\|u_0\|^2_{H^2(\mathbb{R})},\end{aligned}$$

for each $p \in \{1, \infty\}$, $j \in \{0, 1, 2\}$. This proves (4.5). Clearly, estimates (4.4) and (4.5) imply (4.3).

Finally, using the Hölder inequality, (2.4) and (3.2), we obtain

$$\begin{aligned}\int_{\mathbb{R}} |\partial_x u_\varepsilon \partial_x^2 u_\varepsilon| dx &\le \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\ &\le \|u_\varepsilon(t,\cdot)\|^2_{H^2(\mathbb{R})} \le \|u_0\|^2_{H^2(\mathbb{R})},\end{aligned} \tag{4.10}$$

$$\begin{aligned}\|\partial_x^i P_{3,\varepsilon}(t,\cdot)\|_{L^p(\mathbb{R})} &\le 3\left(\left\|\frac{d^{2+i}G_2}{dx^{2+i}}\right\|_{L^p(\mathbb{R})} + \left\|\frac{d^i G_2}{dx^i}\right\|_{L^p(\mathbb{R})}\right)\int_{\mathbb{R}} |\partial_x u_\varepsilon \partial_x^2 u_\varepsilon| dx\\ &\le 6C_0\|u_0\|^2_{H^2(\mathbb{R})},\end{aligned} \tag{4.11}$$

for $p \in \{1, \infty\}$ and $i \in \{0, 1\}$. The estimates (4.4), (4.10), and (4.11) imply (4.6) and (4.7). □

Next we turn to estimates of time derivatives. Introduce the notation

$$\Pi_T := [0,T] \times \mathbb{R},$$

for T positive.

Lemma 4.2. *Assume $k = 2$, (2.9) and (2.15). The following inequalities hold*

$$\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le \frac{1}{\sqrt{2}}\|u_0\|^2_{H^2(\mathbb{R})} + 4C_0\|u_0\|^2_{H^2(\mathbb{R})} + \|u_0\|_{H^2(\mathbb{R})}, \tag{4.12}$$

$$\|\partial_t \partial_x u_\varepsilon\|_{L^2(\Pi_T)} \le \sqrt{2T}\|u_0\|^2_{H^2(\mathbb{R})} + 4C_0\|u_0\|^2_{H^2(\mathbb{R})}\sqrt{T} + \frac{1}{\sqrt{2}}\|u_0\|_{H^2(\mathbb{R})}, \tag{4.13}$$

for each T, $t > 0$ and $0 < \varepsilon < 1$.

Proof. Let T, $t > 0$ and $0 < \varepsilon < 1$. From (2.17) and Lemma 3.2 and 4.1,

$$\begin{aligned}\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} &\le \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} + \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\ &\quad + \varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\ &\le \|u_\varepsilon\|_{L^\infty([0,\infty)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} + \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\ &\quad + \varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\ &\le \frac{1}{\sqrt{2}}\|u_0\|^2_{H^2(\mathbb{R})} + 4C_0\|u_0\|^2_{H^2(\mathbb{R})} + \varepsilon\|u_0\|_{H^2(\mathbb{R})},\end{aligned}$$

this proves (4.12).

Moreover, differentiating (2.17) with respect to x, we get

$$\partial_t\partial_x u_\varepsilon + \big(\partial_x u_\varepsilon\big)^2 + u_\varepsilon\partial_x^2 u_\varepsilon + \partial_x^2 P_\varepsilon = \varepsilon\partial_x^3 u_\varepsilon, \tag{4.14}$$

then, from Lemmas 3.2 and 4.1 we find that

$$\begin{aligned}
&\|\partial_t\partial_x u_\varepsilon\|_{L^2(\Pi_T)}\\
&\quad\le \|\partial_x u_\varepsilon\|^2_{L^4(\Pi_T)} + \|u_\varepsilon\partial_x^2 u_\varepsilon\|_{L^2(\Pi_T)} + \|\partial_x^2 P_\varepsilon\|_{L^2(\Pi_T)} + \varepsilon\|\partial_x^3 u_\varepsilon\|_{L^2(\Pi_T)}\\
&\quad\le \frac{1}{\sqrt{2}}\|u_0\|^2_{H^2(\mathbb{R})}\sqrt{T} + \|u_\varepsilon\|_{L^\infty}\|\partial_x^2 u_\varepsilon\|_{L^2(\Pi_T)} + \|\partial_x^2 P_\varepsilon\|_{L^2(\Pi_T)} + \varepsilon\|\partial_x^3 u_\varepsilon\|_{L^2(\Pi_T)}\\
&\quad\le \sqrt{2T}\|u_0\|^2_{H^2(\mathbb{R})} + 4C_0\|u_0\|^2_{H^2(\mathbb{R})}\sqrt{T} + \frac{1}{\sqrt{2}}\|u_0\|_{H^2(\mathbb{R})},
\end{aligned}$$

which proves (4.13). □

Lemma 4.3. *Assume $k = 2$, (2.9) and (2.15). Let $T > 0$. There exists two positive constants $K_{1,T}$, $K_{2,T}$ depending only on $\|u_0\|_{H^2(\mathbb{R})}$ and T and independent of ε, such that*

$$\|\partial_t\partial_x^3 P_{1,\varepsilon}\|_{L^1(\Pi_T)},\ \|\partial_t\partial_x^3 P_{1,\varepsilon}\|_{L^\infty(\Pi_T)} \le K_{1,T}, \tag{4.15}$$

$$\|\partial_t P_{3,\varepsilon}\|_{L^1(\Pi_T)},\ \|\partial_t P_{3,\varepsilon}\|_{L^\infty(\Pi_T)} \le K_{2,T}, \tag{4.16}$$

for each $0 < \varepsilon < 1$.

Proof. Fix $0 < \varepsilon < 1$ and $T > 0$. We begin by proving (4.15). Observe that

$$\partial_t\partial_x^2 u_\varepsilon + 3\partial_x^2 u_\varepsilon\partial_x u_\varepsilon + u_\varepsilon\partial_x^3 u_\varepsilon + \partial_x^3 P_\varepsilon = \varepsilon\partial_x^4 u_\varepsilon, \tag{4.17}$$

and, from (4.2),

$$\partial_t\partial_x^2 u_\varepsilon + u_\varepsilon\partial_x^3 u_\varepsilon + \partial_x^3 P_{1,\varepsilon} + P_{3,\varepsilon} = \varepsilon\partial_x^4 u_\varepsilon. \tag{4.18}$$

Hence, since G_2 is the Green's function of the operator A_2 (see Remark 2.1), we find from the definition of $P_{1,\varepsilon}$ and (4.18) that

$$\begin{aligned}
\partial_t\partial_x^3 P_{1,\varepsilon}(t,x) &= \int_{\mathbb{R}} G_2'''(x-y)\left[\partial_x u_\varepsilon\partial_t\partial_x u_\varepsilon + 2u_\varepsilon\partial_t u_\varepsilon - \partial_x^2 u_\varepsilon\partial_t\partial_x^2 u_\varepsilon\right]dy\\
&= \int_{\mathbb{R}} G_2'''(x-y)\left[\partial_x u_\varepsilon\partial_t\partial_x u_\varepsilon + 2u_\varepsilon\partial_t u_\varepsilon\right]dy\\
&\quad+ \int_{\mathbb{R}} G_2'''(x-y)\left[\partial_x^2 u_\varepsilon\partial_x^3 P_{1,\varepsilon} + \partial_x^2 u_\varepsilon P_{3,\varepsilon}\right]dy\\
&\quad+ \int_{\mathbb{R}} G_2'''(x-y)\left[u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon - \varepsilon\partial_x^4 u_\varepsilon\partial_x^2 u_\varepsilon\right]dy\\
&= \int_{\mathbb{R}} G_2'''(x-y)\left[\partial_x u_\varepsilon\partial_t\partial_x u_\varepsilon + 2u_\varepsilon\partial_t u_\varepsilon\right]dy\\
&\quad+ \int_{\mathbb{R}} G_2'''(x-y)\left[\partial_x^2 u_\varepsilon\partial_x^3 P_{1,\varepsilon} + \partial_x^2 u_\varepsilon P_{3,\varepsilon}\right]dy\\
&\quad+ \frac{1}{2}u_\varepsilon\big(\partial_x^2 u_\varepsilon\big)^2 - \varepsilon\partial_x^3 u_\varepsilon\partial_x^2 u_\varepsilon\\
&\quad+ \int_{\mathbb{R}} \left(G_2'' - G_2\right)(x-y)\left[\frac{1}{2}u_\varepsilon\big(\partial_x^2 u_\varepsilon\big)^2 - \varepsilon\partial_x^3 u_\varepsilon\partial_x^2 u_\varepsilon\right]dy\\
&\quad- \int_{\mathbb{R}} G_2'''(x-y)\left[\frac{1}{2}\partial_x u_\varepsilon\big(\partial_x^2 u_\varepsilon\big)^2 - \varepsilon\left(\partial_x^3 u_\varepsilon\right)^2\right]dy.
\end{aligned} \tag{4.19}$$

Using the Hölder inequality,

$$\|\partial_t\partial_x^3 P_{1,\varepsilon}\|_{L^1(\Pi_T)} \le C_0\int_{\Pi_T}\Big[|\partial_x u_\varepsilon\partial_t\partial_x u_\varepsilon| + 2|u_\varepsilon\partial_t u_\varepsilon|$$

$$
\begin{aligned}
&+|\partial_x^2 u_\varepsilon \partial_x^3 P_{1,\varepsilon}| + |\partial_x^2 u_\varepsilon P_{3,\varepsilon}| + |u_\varepsilon|\left(\partial_x^2 u_\varepsilon\right)^2 \\
&+2\varepsilon|\partial_x^3 u_\varepsilon \partial_x^2 u_\varepsilon| + \frac{1}{2}|\partial_x u_\varepsilon|\left(\partial_x^2 u_\varepsilon\right)^2 + \varepsilon\left(\partial_x^3 u_\varepsilon\right)^2\Big]dtdx \\
&+\int_{\Pi_T}\left[\frac{1}{2}|u_\varepsilon|\left(\partial_x^2 u_\varepsilon\right)^2 + \varepsilon|\partial_x^3 u_\varepsilon \partial_x^2 u_\varepsilon|\right]dtdx \\
\le{}& C_0\Big[\|\partial_x u_\varepsilon\|_{L^2}\|\partial_t\partial_x u_\varepsilon\|_{L^2} + 2\|u_\varepsilon\|_{L^2}\|\partial_t u_\varepsilon\|_{L^2} \\
&+\|\partial_x^2 u_\varepsilon\|_{L^2}\|\partial_x^3 P_{1,\varepsilon}\|_{L^2} + \|\partial_x^2 u_\varepsilon\|_{L^2}\|P_{3,\varepsilon}\|_{L^2} \\
&+\|u_\varepsilon\|_{L^\infty}\|\partial_x^2 u_\varepsilon\|_{L^2}^2 + 2\varepsilon\|\partial_x^3 u_\varepsilon\|_{L^2}\|\partial_x^2 u_\varepsilon\|_{L^2} \\
&+\frac{1}{2}\|\partial_x u_\varepsilon\|_{L^\infty}\|\partial_x^2 u_\varepsilon\|_{L^2}^2 + \varepsilon\|\partial_x^3 u_\varepsilon\|_{L^2}^2\Big] \\
&+\frac{1}{2}\|u_\varepsilon\|_{L^\infty}\|\partial_x^2 u_\varepsilon\|_{L^2}^2 + \varepsilon\|\partial_x^3 u_\varepsilon\|_{L^2}\|\partial_x^2 u_\varepsilon\|_{L^2}.
\end{aligned}
\tag{4.20}
$$

Then, the estimate (4.15) is consequence of (2.4), (3.1), (3.2), (4.4), (4.7), (4.12), and (4.13).

We continue by proving (4.16). Observing that,

$$
\begin{aligned}
P_{3,\varepsilon}(t,x) &= -\frac{3}{2}\int_{\mathbb{R}}\left(G_2'''(x-y) - G_2'(x-y)\right)\left(\partial_x u_\varepsilon(t,y)\right)^2 dy, \\
\partial_t P_{3,\varepsilon}(t,x) &= -3\int_{\mathbb{R}}\left(G_2'''(x-y) - G_2'(x-y)\right)\partial_x u_\varepsilon(t,y)\partial_t\partial_x u_\varepsilon(t,y)dy,
\end{aligned}
$$

we have

$$
\begin{aligned}
\|\partial_t P_{3,\varepsilon}\|_{L^p(\Pi_T)} &\le 3\left(\|G_2'''\|_{L^p(\mathbb{R})} + \|G_2'\|_{L^p(\mathbb{R})}\right)\int_{\Pi_T}|\partial_x u_\varepsilon \partial_t\partial_x u_\varepsilon|\,dx \\
&\le 3\left(\|G_2'''\|_{L^p(\mathbb{R})} + \|G_2'\|_{L^p(\mathbb{R})}\right)\|\partial_x u_\varepsilon\|_{L^2(\Pi_T)}\|\partial_t\partial_x u_\varepsilon\|_{L^2(\Pi_T)},
\end{aligned}
\tag{4.21}
$$

for $p \in \{1, \infty\}$. Hence, the estimate (4.16) follows from (2.4), (3.1) and (4.13). □

Now we look for invariance properties of the problem (2.17).

Lemma 4.4. *Assume $k = 2$, (2.9) and (2.15). The following estimate holds*

$$
\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R})} \le \|\partial_x^2 u_{0,\varepsilon}\|_{L^p(\mathbb{R})}e^{K_1 t} + K_2\frac{e^{K_1 t} - 1}{K_1},
\tag{4.22}
$$

for each $t \ge 0$, $2 \le p < \infty$ and $\varepsilon > 0$, where

$$
K_1 := \frac{1}{p\sqrt{2}}\|u_0\|_{H^2(\mathbb{R})}, \qquad K_2 := (18C_0 + 1)^2\|u_0\|_{H^2(\mathbb{R})}^6.
$$

Proof. Let $2 \le p < \infty$. Denote

$$
q_\varepsilon := \partial_x^2 u_\varepsilon,
$$

there results

$$
\partial_t q_\varepsilon + 3q_\varepsilon\partial_x u_\varepsilon + u_\varepsilon\partial_x q_\varepsilon + \partial_x^3 P_\varepsilon = \varepsilon\partial_x^2 q_\varepsilon,
\tag{4.23}
$$

and, from (4.2),

$$
\partial_t q_\varepsilon + u_\varepsilon\partial_x q_\varepsilon + \widetilde{P}_\varepsilon = \varepsilon\partial_x^2 q_\varepsilon,
\tag{4.24}
$$

where

$$
\widetilde{P}_\varepsilon := \partial_x^3 P_{1,\varepsilon} + P_{3,\varepsilon}.
\tag{4.25}
$$

Multiplying (4.24) by $pq_\varepsilon|q_\varepsilon|^{p-2}$ there results

$$
\begin{aligned}
\partial_t(|q_\varepsilon|^p) + u_\varepsilon\partial_x(|q_\varepsilon|^p) + p\widetilde{P}_\varepsilon q_\varepsilon|q_\varepsilon|^{p-2} &= p\varepsilon q_\varepsilon|q_\varepsilon|^{p-2}\partial_x^2 q_\varepsilon \\
&= \varepsilon\partial_x^2(|q_\varepsilon^2|) - \varepsilon p(p-1)(\partial_x q_\varepsilon)^2.
\end{aligned}
\tag{4.26}
$$

By (3.2), (4.4), and (4.7),

$$\begin{aligned}
p\left\|q_\varepsilon(t,\cdot)\right\|_{L^p(\mathbb{R})}^{p-1}\frac{d}{dt}\left\|q_\varepsilon(t,\cdot)\right\|_{L^p(\mathbb{R})}&=\frac{d}{dt}\int_{\mathbb{R}}|q_\varepsilon|^p dx\\
&\le\int_{\mathbb{R}}\partial_x u_\varepsilon|q_\varepsilon|^p dx+p\int_{\mathbb{R}}|\widetilde{P}_\varepsilon||q_\varepsilon|^{p-1}dx\\
&\le K_1\int_{\mathbb{R}}|q_\varepsilon|^p dx+p\left\|\widetilde{P}_\varepsilon(t,\cdot)\right\|_{L^p(\mathbb{R})}\left\|q_\varepsilon(t,\cdot)\right\|_{L^p(\mathbb{R})}^{p-1}\\
&\le K_1\left\|q_\varepsilon(t,\cdot)\right\|_{L^p(\mathbb{R})}^{p}+pK_2\left\|q_\varepsilon(t,\cdot)\right\|_{L^p(\mathbb{R})}^{p-1},
\end{aligned}$$

hence

$$\frac{d}{dt}\left\|q_\varepsilon(t,\cdot)\right\|_{L^p(\mathbb{R})}\le\frac{K_1}{p}\left\|q_\varepsilon(t,\cdot)\right\|_{L^p(\mathbb{R})}+K_2.$$

The claim is a direct consequence of the Gronwall inequality. ☐

Lemma 4.5. *Assume $k=2$, (2.9) and (2.15). The following estimate holds*

$$\begin{aligned}
\|\partial_x^3u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}&+2\varepsilon\int_0^t e^{K_3(t-\tau)}\|\partial_x^4u_\varepsilon(\tau,\cdot)\|^2_{L^2(\mathbb{R})}d\tau\\
&\le\|\partial_x^3u_{0,\varepsilon}\|^2_{L^2(\mathbb{R})}e^{K_3t}+K_4\frac{e^{K_3t}-1}{K_3},
\end{aligned}\tag{4.27}$$

for each $t\ge0$ and $\varepsilon>0$, where

$$K_3:=\frac{1}{\sqrt{2}}\|u_0\|_{H^2(\mathbb{R})}+\frac{7}{2},\qquad K_4:=\left(\frac{3}{4}+16C_0^2\right)\|u_0\|^4_{H^2(\mathbb{R})}.$$

Proof. Using the notation from the proof of Lemma 4.4, we have

$$\partial_t\partial_xq_\varepsilon+\partial_xu_\varepsilon\partial_xq_\varepsilon-\frac12q_\varepsilon^2+u_\varepsilon\partial_x^2q_\varepsilon+\frac12\big(\partial_xu_\varepsilon\big)^2+u_\varepsilon^2+\partial_x^2P_\varepsilon-P_\varepsilon=\varepsilon\partial_x^3q_\varepsilon.\tag{4.28}$$

By (4.28), (3.2), and (4.3),

$$\begin{aligned}
\frac12\frac{d}{dt}\int_{\mathbb{R}}\left(\partial_xq_\varepsilon\right)^2dx&=\int_{\mathbb{R}}\partial_xq_\varepsilon\partial_t\partial_xq_\varepsilon dx\\
&=\varepsilon\int_{\mathbb{R}}\partial_x^3q_\varepsilon\partial_xq_\varepsilon dx-\int_{\mathbb{R}}\partial_xu_\varepsilon\left(\partial_xq_\varepsilon\right)^2dx\\
&\quad+\frac12\int_{\mathbb{R}}q_\varepsilon^2\partial_xq_\varepsilon dx-\int_{\mathbb{R}}u_\varepsilon\partial_x^2q_\varepsilon\partial_xq_\varepsilon dx\\
&\quad-\frac12\int_{\mathbb{R}}\big(\partial_xu_\varepsilon\big)^2\partial_xq_\varepsilon dx-\int_{\mathbb{R}}u_\varepsilon^2\partial_xq_\varepsilon dx\\
&\quad-\int_{\mathbb{R}}\partial_x^2P_\varepsilon\partial_xq_\varepsilon dx+\int_{\mathbb{R}}P_\varepsilon\partial_xq_\varepsilon dx\\
&=-\varepsilon\int_{\mathbb{R}}\big(\partial_x^2q_\varepsilon\big)^2dx-\frac12\int_{\mathbb{R}}\partial_xu_\varepsilon\left(\partial_xq_\varepsilon\right)^2dx\\
&\quad-\frac12\int_{\mathbb{R}}\partial_xq_\varepsilon\big(\partial_xu_\varepsilon\big)^2dx-\int_{\mathbb{R}}u_\varepsilon^2\partial_xq_\varepsilon dx\\
&\quad-\int_{\mathbb{R}}\partial_x^2P_\varepsilon\partial_xq_\varepsilon dx+\int_{\mathbb{R}}P_\varepsilon\partial_xq_\varepsilon dx\\
&\le-\varepsilon\int_{\mathbb{R}}\big(\partial_x^2q_\varepsilon\big)^2dx+\left(\frac12\|\partial_xu_\varepsilon\|_{L^\infty([0,\infty)\times\mathbb{R})}+\frac74\right)\int_{\mathbb{R}}\left(\partial_xq_\varepsilon\right)^2dx\\
&\quad+\frac14\int_{\mathbb{R}}\big(\partial_xu_\varepsilon\big)^4dx+\frac12\int_{\mathbb{R}}u_\varepsilon^4dx+\frac12\int_{\mathbb{R}}\big(\partial_x^2P_\varepsilon\big)^2dx+\frac12\int_{\mathbb{R}}P_\varepsilon^2dx\\
&\le-\varepsilon\int_{\mathbb{R}}\big(\partial_x^2q_\varepsilon\big)^2dx+\frac{K_3}{2}\int_{\mathbb{R}}\left(\partial_xq_\varepsilon\right)^2dx+\frac{K_4}{2},
\end{aligned}$$

hence, using the Gronwall inequality, we get (4.27). □

Remark 4.6. *Assuming* $k = 2$, (2.9) *and* (2.15). *From* (4.24), (4.27) *and Lemmas* 3.2 *and* 4.1, *we get*

$$\|\partial_t \partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le K_5 e^{K_5 t}(1+t)\|u_{0,\varepsilon}\|_{H^3(\mathbb{R})}, \tag{4.29}$$

$$\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le M_\varepsilon(t), \tag{4.30}$$

for each $t \ge 0$ *and* $\varepsilon > 0$, *where* $K_5 > 0$ *is a constant depending only on* $\|u_0\|_{H^2(\mathbb{R})}$ *but independent of* ε, *and*

$$M_\varepsilon(t)^2 := \frac{1}{2}\|u_0\|_{H^2(\mathbb{R})} + \frac{1}{2}\|\partial_x^3 u_{0,\varepsilon}\|^2_{L^2(\mathbb{R})} e^{K_3 t} + K_4 \frac{e^{K_3 t} - 1}{2K_3}.$$

Lemma 4.7. *Assume* $k = 2$, (2.9) *and* (2.15). *There exists a constant* $K_6 > 0$ *depending only on* $\|u_0\|_{H^2(\mathbb{R})}$ *but independent of* ε, *such that*

$$\|\partial_t \partial_x^i P_\varepsilon\|_{L^1(\Pi_T)} \le K_6 e^{K_6 T}(1+T)\|u_{0,\varepsilon}\|_{H^3(\mathbb{R})}, \tag{4.31}$$

for each $i \in \{0, 1, 2\}$, $T \ge 0$ *and* $\varepsilon > 0$.

Proof. Using Remark 2.1, we know

$$P_\varepsilon = P_{4,\varepsilon} + P_{5,\varepsilon}, \tag{4.32}$$

where

$$P_{4,\varepsilon}(t,x) := \int_{\mathbb{R}} G_2(x-y)\left[\frac{1}{2}\left(\partial_x u_\varepsilon(t,y)\right)^2 + u_\varepsilon^2(t,y)\right]dy,$$
$$P_{5,\varepsilon}(t,x) := -\int_{\mathbb{R}} G_2(x-y)\left[\frac{7}{2}\left(\partial_x^2 u_\varepsilon(t,y)\right)^2 + 3\partial_x u_\varepsilon(t,y)\partial_x^3 u_\varepsilon(t,y)\right]dy.$$

Observe that, for each $i \in \{0, 1, 2\}$,

$$\partial_t \partial_x^i P_{4,\varepsilon}(t,x) = \int_{\mathbb{R}} \frac{d^i G_2}{dx^i}(x-y)\left[\partial_x u_\varepsilon(t,y)\partial_t\partial_x u_\varepsilon(t,y) + 2u_\varepsilon(t,y)\partial_t u_\varepsilon(t,y)\right]dy,$$

then, by (2.4), (3.2), and Remark 4.2,

$$\begin{aligned}
&\|\partial_t \partial_x^i P_{4,\varepsilon}\|_{L^1(\Pi_T)} \\
&\quad\le \int_{\Pi_T\times\mathbb{R}} \left|\frac{d^i G_2}{dx^i}(x-y)\right| |\partial_x u_\varepsilon \partial_t\partial_x u_\varepsilon + 2u_\varepsilon\partial_t u_\varepsilon| dtdxdy \\
&\quad\le C_0 \int_{\Pi_T} \left(|\partial_x u_\varepsilon||\partial_t\partial_x u_\varepsilon| + 2|u_\varepsilon||\partial_t u_\varepsilon|\right)dtdx \\
&\quad\le C_0\left(\|\partial_x u_\varepsilon\|_{L^2(\Pi_T)}\|\partial_t\partial_x u_\varepsilon\|_{L^2(\Pi_T)} + 2\|u_\varepsilon\|_{L^2(\Pi_T)}\|\partial_t u_\varepsilon\|_{L^2(\Pi_T)}\right) \\
&\quad\le c_1(1+T),
\end{aligned} \tag{4.33}$$

for some constant $c_1 > 0$ depending only on $\|u_0\|_{H^2(\mathbb{R})}$.

Moreover,

$$\begin{aligned}
\partial_t P_{5,\varepsilon}(t,x) &= -\int_{\mathbb{R}} G_2(x-y)\left[7\partial_x^2 u_\varepsilon \partial_t\partial_x^2 u_\varepsilon + 3\partial_t\partial_x u_\varepsilon \partial_x^3 u_\varepsilon + 3\partial_x u_\varepsilon \partial_t \partial_x^3 u_\varepsilon\right]dy \\
&= 4\int_{\mathbb{R}} G_2(x-y)\partial_t\partial_x u_\varepsilon \partial_x^3 u_\varepsilon dy - 7\int_{\mathbb{R}} G_2'(x-y)\partial_t\partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \\
&\quad - 3\int_{\mathbb{R}} G_2'(x-y)\partial_t\partial_x^2 u_\varepsilon \partial_x u_\varepsilon dy + 3\int_{\mathbb{R}} G_2(x-y)\partial_t\partial_x^2 u_\varepsilon \partial_x^2 u_\varepsilon dy \\
&= \int_{\mathbb{R}} G_2(x-y)\partial_t\partial_x u_\varepsilon \partial_x^3 u_\varepsilon dy - \int_{\mathbb{R}} G_2'(x-y)\partial_t\partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \\
&\quad - 3\int_{\mathbb{R}} G_2''(x-y)\partial_t\partial_x u_\varepsilon \partial_x u_\varepsilon dy,
\end{aligned} \tag{4.34}$$

then, by (2.4), (3.2), (4.27), and Remark 4.2, using the same argument as for (4.33), for $i \in \{0, 1\}$ we infer

$$
\begin{aligned}
\|\partial_t\partial_x^{i+1}P_{5,\varepsilon}\|_{L^1(\Pi_T)} \leq &\int_{\Pi_T\times\mathbb{R}} \left|\frac{d^iG_2}{dx^i}(x-y)\right| |\partial_t\partial_x u_\varepsilon \partial_x^3 u_\varepsilon| dtdxdy\\
&+\int_{\Pi_T\times\mathbb{R}} \left|\frac{d^{i+1}G_2}{dx^{i+1}}(x-y)\right| |\partial_t\partial_x u_\varepsilon \partial_x^2 u_\varepsilon| dtdxdy\\
&+3\int_{\Pi_T\times\mathbb{R}} \left|\frac{d^{i+2}G_2}{dx^{i+2}}(x-y)\right| |\partial_t\partial_x u_\varepsilon \partial_x u_\varepsilon| dtdxdy\\
\leq& 3C_0\|\partial_t\partial_x u_\varepsilon\|_{L^2(\Pi_T)}\|u_\varepsilon\|_{H^3(\Pi_T)}\\
\leq& c_2e^{c_2T}(1+T)\|u_{0,\varepsilon}\|_{H^3(\mathbb{R})},
\end{aligned}
\tag{4.35}
$$

for some constant $c_2 > 0$ depending only on $\|u_0\|_{H^2(\mathbb{R})}$.

Since G_2 is the Green's function of A_2, we have

$$
\begin{aligned}
\partial_t\partial_x^2P_{5,\varepsilon}(t,x) =& \int_{\mathbb{R}} G_2''(x-y)\partial_t\partial_x u_\varepsilon\partial_x^3u_\varepsilon dy - \int_{\mathbb{R}} G_2'''(x-y)\partial_t\partial_x u_\varepsilon\partial_x^2u_\varepsilon dy\\
&-3\partial_t\partial_x u_\varepsilon(t,x)\partial_x u_\varepsilon(t,x) - 3\int_{\mathbb{R}} G_2''(x-y)\partial_t\partial_x u_\varepsilon\partial_x u_\varepsilon dy\\
&+3\int_{\mathbb{R}} G_2(x-y)\partial_t\partial_x u_\varepsilon\partial_x u_\varepsilon dy,
\end{aligned}
\tag{4.36}
$$

then, by (2.4), (3.2), (4.27), and Remark 4.2, using the same argument of (4.33),

$$
\begin{aligned}
\|\partial_t\partial_x^2P_{5,\varepsilon}\|_{L^1(\Pi_T)} \leq &\int_{\Pi_T\times\mathbb{R}} |G_2''(x-y)||\partial_t\partial_x u_\varepsilon\partial_x^3u_\varepsilon| dtdxdy\\
&+\int_{\Pi_T\times\mathbb{R}} |G_2'''(x-y)||\partial_t\partial_x u_\varepsilon\partial_x^2u_\varepsilon| dtdxdy\\
&+3\int_{\Pi_T} |\partial_t\partial_x u_\varepsilon\partial_x u_\varepsilon| dtdx\\
&+3\int_{\Pi_T\times\mathbb{R}} |G_2''(x-y)||\partial_t\partial_x u_\varepsilon\partial_x u_\varepsilon| dtdxdy\\
&+3\int_{\Pi_T\times\mathbb{R}} G_2(x-y)|\partial_t\partial_x u_\varepsilon\partial_x u_\varepsilon| dtdxdy\\
\leq& 6C_0\|\partial_t\partial_x u_\varepsilon\|_{L^2(\Pi_T)}\|u_\varepsilon\|_{H^3(\Pi_T)}\\
\leq& c_3e^{c_3T}(1+T)\|u_{0,\varepsilon}\|_{H^3(\mathbb{R})},
\end{aligned}
$$

for some constant $c_3 > 0$ depending only on $\|u_0\|_{H^2(\mathbb{R})}$. □

5. Stability with respect to viscosity and initial data

In this section we prove a stability estimate for (2.17) with respect to the viscosity coefficient ε and the initial condition u_0. We only consider the case $k = 2$, and refer to Appendix B for the general case $k > 2$.

Let $v, w \in C([0,\infty); H^3(\mathbb{R}))$ be the solutions of (see Lemma 3.1)

$$
\begin{cases}
\partial_t v + v\partial_x v + \partial_x V = \lambda\partial_x^2 v,\\
\partial_x^4 V - \partial_x^2 V + V = \frac{1}{2}(\partial_x v)^2 - \frac{7}{2}\left(\partial_x^2 v\right)^2 - 3\partial_x v\partial_x^3 v + v^2,\\
v(0,x) = v_0(x),
\end{cases}
\tag{5.1}
$$

and

$$\begin{cases} \partial_t w + w\partial_x w + \ \partial_x W = \mu\partial_x^2 w, \\ \partial_x^4 W - \partial_x^2 W + W = \dfrac{1}{2}\left(\partial_x w\right)^2 - \dfrac{7}{2}\left(\partial_x^2 w\right)^2 - 3\partial_x w\partial_x^3 w + w^2, \\ w(0,x) = w_0(x), \end{cases} \tag{5.2}$$

respectively, where we assume

$$\lambda,\, \mu > 0, \qquad v_0,\, w_0 \in H^3(\mathbb{R}). \tag{5.3}$$

From Lemma 3.2 we know that

$$\|v(t,\,\cdot\,)\|_{H^2(\mathbb{R})} \le \|v_0\|_{H^2(\mathbb{R})}, \tag{5.4}$$

$$\|v\|_{L^\infty([0,\infty)\times\mathbb{R})}, \|\partial_x v\|_{L^\infty([0,\infty)\times\mathbb{R})} \le \frac{1}{\sqrt{2}}\|v_0\|_{H^2(\mathbb{R})}, \tag{5.5}$$

$$\|\partial_x^3 v(t,\,\cdot\,)\|^2_{L^2(\mathbb{R})} \le \|\partial_x^3 v_0\|^2_{L^2(\mathbb{R})} e^{C_1 t} + C_2\frac{e^{C_1 t}-1}{C_1}, \tag{5.6}$$

$$\|w(t,\,\cdot\,)\|_{H^2(\mathbb{R})} \le \|w_0\|_{H^2(\mathbb{R})}, \tag{5.7}$$

$$\|w\|_{L^\infty([0,\infty)\times\mathbb{R})}, \|\partial_x w\|_{L^\infty([0,\infty)\times\mathbb{R})} \le \frac{1}{\sqrt{2}}\|w_0\|_{H^2(\mathbb{R})}, \tag{5.8}$$

$$\|\partial_x^3 w(t,\,\cdot\,)\|^2_{L^2(\mathbb{R})} \le \|\partial_x^3 w_0\|^2_{L^2(\mathbb{R})} e^{C_3 t} + C_4\frac{e^{C_3 t}-1}{C_3}, \tag{5.9}$$

for each $t \ge 0$, where

$$C_1 := \frac{1}{\sqrt{2}}\|v_0\|_{H^2(\mathbb{R})} + \frac{7}{2}, \qquad C_2 := \left(\frac{3}{4} + 16C_0^2\right)\|v_0\|^4_{H^2(\mathbb{R})},$$
$$C_3 := \frac{1}{\sqrt{2}}\|w_0\|_{H^2(\mathbb{R})} + \frac{7}{2}, \qquad C_4 := \left(\frac{3}{4} + 16C_0^2\right)\|w_0\|^4_{H^2(\mathbb{R})}.$$

The main result of this section is the following.

Theorem 5.1. *Assume* (5.3). *There results*

$$\|v(t,\,\cdot\,) - w(t,\,\cdot\,)\|_{H^2(\mathbb{R})} \le e^{At/2}\|w_0 - v_0\|_{H^2(\mathbb{R})} + B_T|\mu-\lambda|\left(\frac{e^{At}-1}{A}\right)^{1/2}, \tag{5.10}$$

for each $0 \le t \le T$ *and* $0 \le \theta \le 1$, *where*

$$A := \left(\|v_0\|_{H^2(\mathbb{R})} + \|w_0\|_{H^2(\mathbb{R})}\right)\left[2\sqrt{2} + \frac{1}{\sqrt{2}} + \frac{\|v_0\|_{H^2(\mathbb{R})} + \|w_0\|_{H^2(\mathbb{R})}}{2\min\{\lambda,\mu\}}\right],$$
$$B_T^2 := \frac{1}{2}(\|v_0\|_{H^2(\mathbb{R})} + \|w_0\|_{H^2(\mathbb{R})})^2$$
$$+ \left(\|\partial_x^3 v_0\|_{L^2(\mathbb{R})} + \|\partial_x^3 w_0\|_{L^2(\mathbb{R})}\right)^2 \frac{e^{C_5 T}}{4\min\{\lambda,\mu\}} + \frac{C_6}{4\min\{\lambda,\mu\}}\frac{e^{C_5 T}-1}{C_5},$$
$$C_5 := \frac{1}{\sqrt{2}}\left(\|v_0\|_{H^2(\mathbb{R})} + \|w_0\|_{H^2(\mathbb{R})}\right) + \frac{7}{2},$$
$$C_6 := \left(\frac{3}{4} + 16C_0^2\right)\left(\|v_0\|_{H^2(\mathbb{R})} + \|w_0\|_{H^2(\mathbb{R})}\right)^4,$$

for every $T \ge 0$.

Our approach, as in [3, 4], is based on the following homotopy argument. Let $0 \le \theta \le 1$. The function ω_θ interpolates between the functions v and w. More precisely, denote by ω_θ the solution of the initial value problem (see Lemma 3.1)

$$\begin{cases}\partial_t\omega_\theta+\omega_\theta\partial_x\omega_\theta+\partial_x\Omega_\theta=(\theta\mu+(1-\theta)\lambda)\partial_x^2\omega_\theta,\\ \partial_x^4\Omega_\theta-\partial_x^2\Omega_\theta+\Omega_\theta=\dfrac{1}{2}\left(\partial_x\omega_\theta\right)^2-\dfrac{7}{2}\left(\partial_x^2\omega_\theta\right)^2-3\partial_x\omega_\theta\partial_x^3\omega_\theta+\omega_\theta^2,\\ \omega_\theta(0,x)=\theta w_0(x)+(1-\theta)v_0(x).\end{cases}\tag{5.11}$$

Clearly

$$\omega_0=v,\qquad \omega_1=w.$$

Indeed

$$\theta\mapsto\omega_\theta(t,x)$$

is a curve joining $v(t,x)$ and $w(t,x)$, and

$$\begin{aligned}\|v(t,\cdot)-w(t,\cdot)\|_{H^2(\mathbb{R})}&\equiv\mathrm{dist}_{H^2(\mathbb{R})}\big(v(t,\cdot),w(t,\cdot)\big)\\ &\le\mathrm{length}_{H^2(\mathbb{R})}\big(\omega_\theta(t,\cdot)\big),\end{aligned}\tag{5.12}$$

for each $t\ge 0$.

Remark 5.2. *From Lemma* 3.2 *and* (5.4)*,* (5.5)*,* (5.6)*,* (5.7)*,* (5.8)*, and* (5.9)*, we know that*

$$\|\omega_\theta(t,\cdot)\|_{H^2(\mathbb{R})}\le\|v_0\|_{H^2(\mathbb{R})}+\|w_0\|_{H^2(\mathbb{R})},\tag{5.13}$$

$$\|\omega_\theta\|_{L^\infty([0,\infty)\times\mathbb{R})},\|\partial_x\omega_\theta\|_{L^\infty([0,\infty)\times\mathbb{R})}\le\frac{1}{\sqrt{2}}\big(\|v_0\|_{H^2(\mathbb{R})}+\|w_0\|_{H^2(\mathbb{R})}\big),\tag{5.14}$$

$$\|\partial_x^3\omega_\theta(t,\cdot)\|^2_{L^2(\mathbb{R})}\le\big(\|\partial_x^3v_0\|_{L^2(\mathbb{R})}+\|\partial_x^3w_0\|_{L^2(\mathbb{R})}\big)^2e^{C_5t}+C_6\frac{e^{C_5t}-1}{C_5},\tag{5.15}$$

for each $t\ge 0$ and $0\le\theta\le 1$.

Arguing as in [3, Lemma 3.2], [4, Lemma 3.2], we have the following result.

Lemma 5.3 (Smoothness of $\theta\mapsto\omega_\theta$). *Assume* (5.3)*. The curve*

$$\theta\in[0,1]\mapsto\omega_\theta(t,\cdot)\in C^2(\mathbb{R})$$

is of class C^1. In particular, we infer

$$\mathrm{length}_{H^2(\mathbb{R})}\big(\omega_\theta(t,\cdot)\big)=\int_0^1\|\partial_\theta\omega_\theta(t,\cdot)\|_{H^2(\mathbb{R})}d\theta,\tag{5.16}$$

for each $t\ge 0$.

Denoting

$$z_\theta:=\partial_\theta\omega_\theta,\qquad Z_\theta:=\partial_\theta\Omega_\theta,$$

and differentiating the equations in (5.11) with respect to θ, we have

$$\begin{cases}\partial_tz_\theta+z_\theta\partial_x\omega_\theta+\omega_\theta\partial_xz_\theta+\partial_xZ_\theta=(\theta\mu+(1-\theta)\lambda)\partial_x^2z_\theta+(\mu-\lambda)\partial_x^2\omega_\theta,\\ A_2(Z_\theta)=\partial_x\omega_\theta\partial_xz_\theta-7\partial_x^2\omega_\theta\partial_x^2z_\theta-3\partial_xz_\theta\partial_x^3\omega_\theta-3\partial_x\omega_\theta\partial_x^3z_\theta+2\omega_\theta z_\theta,\\ z_\theta(0,x)=w_0(x)-v_0(x).\end{cases}\tag{5.17}$$

We now turn to the proof of Theorem 5.1. The following lemma is needed.

Lemma 5.4. *Assume* (5.3)*. There results*

$$\|z_\theta(t,\cdot)\|_{H^2(\mathbb{R})}\le e^{A_Tt/2}\|w_0-v_0\|_{H^2(\mathbb{R})}+B_T|\mu-\lambda|\left(\frac{e^{At}-1}{A}\right)^{1/2},\tag{5.18}$$

for each $0\le t\le T$ and $0\le\theta\le 1$.

Proof. Let $0<\theta<1$ and $t>0$. By (5.17),

$$
\begin{aligned}
(5.19)\qquad \frac{1}{2}\frac{d}{dt}\|z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})} &= \int_{\mathbb{R}}\partial_t z_\theta A_2(z_\theta)dx\\
&=(\theta\mu+(1-\theta)\lambda)\int_{\mathbb{R}}\partial_x^2 z_\theta A_2(z_\theta)dx+(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta A_2(z_\theta)dx\\
&\quad-\int_{\mathbb{R}}z_\theta\partial_x\omega_\theta A_2(z_\theta)dx-\int_{\mathbb{R}}\omega_\theta\partial_x z_\theta A_2(z_\theta)dx-\int_{\mathbb{R}}\partial_x Z_\theta A_2(z_\theta)dx\\
&=-(\theta\mu+(1-\theta)\lambda)\|\partial_x z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})}+(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta z_\theta dx\\
&\quad-(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta\partial_x^2 z_\theta dx+(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta\partial_x^4 z_\theta dx\\
&\quad-\int_{\mathbb{R}}z_\theta^2\partial_x\omega_\theta dx+\int_{\mathbb{R}}z_\theta\partial_x\omega_\theta\partial_x^2 z_\theta dx-\int_{\mathbb{R}}z_\theta\partial_x\omega_\theta\partial_x^4 z_\theta dx\\
&\quad-\int_{\mathbb{R}}\omega_\theta z_\theta\partial_x z_\theta dx+\int_{\mathbb{R}}\omega_\theta\partial_x z_\theta\partial_x^2 z_\theta dx-\int_{\mathbb{R}}\omega_\theta\partial_x z_\theta\partial_x^4 z_\theta dx\\
&\quad+\int_{\mathbb{R}}A_2(Z_\theta)\partial_x z_\theta dx\\
&=-(\theta\mu+(1-\theta)\lambda)\|\partial_x z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})}+(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta z_\theta dx\\
&\quad-(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta\partial_x^2 z_\theta dx+(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta\partial_x^4 z_\theta dx\\
&\quad-\int_{\mathbb{R}}z_\theta^2\partial_x\omega_\theta dx+\int_{\mathbb{R}}z_\theta\partial_x\omega_\theta\partial_x^2 z_\theta dx-\int_{\mathbb{R}}z_\theta\partial_x\omega_\theta\partial_x^4 z_\theta dx\\
&\quad+\int_{\mathbb{R}}\omega_\theta\partial_x z_\theta\partial_x^2 z_\theta dx-\int_{\mathbb{R}}\omega_\theta\partial_x z_\theta\partial_x^4 z_\theta dx\\
&\quad+\int_{\mathbb{R}}\partial_x\omega_\theta\left(\partial_x z_\theta\right)^2dx-7\int_{\mathbb{R}}\partial_x^2\omega_\theta\partial_x^2 z_\theta\partial_x z_\theta dx\\
&\quad-3\int_{\mathbb{R}}\partial_x^3\omega_\theta\left(\partial_x z_\theta\right)^2dx-3\int_{\mathbb{R}}\partial_x\omega_\theta\partial_x^3 z_\theta\partial_x z_\theta dx+\int_{\mathbb{R}}\omega_\theta z_\theta\partial_x z_\theta dx\\
&=-(\theta\mu+(1-\theta)\lambda)\|\partial_x z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})}+(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta z_\theta dx\\
&\quad-(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta\partial_x^2 z_\theta dx-(\mu-\lambda)\int_{\mathbb{R}}\partial_x^3\omega_\theta\partial_x^3 z_\theta dx\\
&\quad-\int_{\mathbb{R}}z_\theta^2\partial_x\omega_\theta dx+\int_{\mathbb{R}}z_\theta\partial_x\omega_\theta\partial_x^2 z_\theta dx+\int_{\mathbb{R}}z_\theta\partial_x^2\omega_\theta\partial_x^3 z_\theta dx\\
&\quad+\int_{\mathbb{R}}\omega_\theta\partial_x z_\theta\partial_x^2 z_\theta dx+\int_{\mathbb{R}}\omega_\theta\partial_x^2 z_\theta\partial_x^3 z_\theta dx+\int_{\mathbb{R}}\partial_x\omega_\theta\left(\partial_x z_\theta\right)^2dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}}\partial_x^3\omega_\theta\left(\partial_x z_\theta\right)^2dx-\int_{\mathbb{R}}\partial_x\omega_\theta\partial_x z_\theta\partial_x^3 z_\theta dx+\int_{\mathbb{R}}\omega_\theta z_\theta\partial_x z_\theta dx\\
&=I_1(t)+I_2(t)+I_3(t)+I_4(t),
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(t)&:=(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta z_\theta dx-(\mu-\lambda)\int_{\mathbb{R}}\partial_x^2\omega_\theta\partial_x^2 z_\theta dx,\\
I_2(t)&:=-\int_{\mathbb{R}}z_\theta^2\partial_x\omega_\theta dx+\int_{\mathbb{R}}z_\theta\partial_x\omega_\theta\partial_x^2 z_\theta dx\\
&\quad+\int_{\mathbb{R}}\omega_\theta\partial_x z_\theta\partial_x^2 z_\theta dx+\int_{\mathbb{R}}\partial_x\omega_\theta\left(\partial_x z_\theta\right)^2dx+\int_{\mathbb{R}}\omega_\theta z_\theta\partial_x z_\theta dx,
\end{aligned}
$$

$$
\begin{aligned}
I_3(t) &:= -\int_{\mathbb{R}} \partial_x\omega_\theta \partial_x z_\theta \partial_x^3 z_\theta dx + \int_{\mathbb{R}} \omega_\theta \partial_x^2 z_\theta \partial_x^3 z_\theta dx + \frac{1}{2}\int_{\mathbb{R}} \partial_x^3\omega_\theta \left(\partial_x z_\theta\right)^2 dx,\\
I_4(t) &:= -(\theta\mu + (1-\theta)\lambda)\|\partial_x z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})} - (\mu-\lambda)\int_{\mathbb{R}} \partial_x^3\omega_\theta \partial_x^3 z_\theta dx\\
&\quad + \int_{\mathbb{R}} z_\theta \partial_x^2\omega_\theta \partial_x^3 z_\theta dx.
\end{aligned}
$$

We can estimate I_1 in the following way. From (5.13),

$$
\begin{aligned}
|I_1(t)| &\le |\mu-\lambda|^2 \int_{\mathbb{R}} \left(\partial_x^2\omega_\theta\right)^2 dx + \frac{1}{2}\int_{\mathbb{R}} z_\theta^2 dx + \frac{1}{2}\int_{\mathbb{R}} \left(\partial_x^2 z_\theta\right)^2 dx\\
&\le |\mu-\lambda|^2\left(\|v_0\|_{H^2(\mathbb{R})} + \|w_0\|_{H^2(\mathbb{R})}\right)^2 + \frac{1}{2}\|z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})}.
\end{aligned}
\tag{5.20}
$$

Furthermore, I_2 can be estimated as follows. From (5.14),

$$
\begin{aligned}
|I_2(t)| &\le \frac{1}{2}\left(\|\omega_\theta\|_{L^\infty} + 3\|\partial_x\omega_\theta\|_{L^\infty}\right)\int_{\mathbb{R}} z_\theta^2 dx\\
&\quad + \left(\|\omega_\theta\|_{L^\infty} + \|\partial_x\omega_\theta\|_{L^\infty}\right)\int_{\mathbb{R}} \left(\partial_x z_\theta\right)^2 dx\\
&\quad + \frac{1}{2}\left(\|\omega_\theta\|_{L^\infty} + 3\|\partial_x\omega_\theta\|_{L^\infty}\right)\int_{\mathbb{R}} \left(\partial_x^2 z_\theta\right)^2 dx\\
&\le \sqrt{2}\left(\|v_0\|_{H^2(\mathbb{R})} + \|w_0\|_{H^2(\mathbb{R})}\right)\|z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})}.
\end{aligned}
\tag{5.21}
$$

Integrating by parts we find

$$
I_3(t) = \frac{1}{2}\int_{\mathbb{R}} \partial_x\omega_\theta \left(\partial_x^2 z_\theta\right)^2 dx,
$$

so, by (5.14), (5.15),

$$
|I_3(t)| \le \frac{1}{2}\|\partial_x\omega_\theta\|_{L^\infty}\int_{\mathbb{R}} \left(\partial_x^2 z_\theta\right)^2 dx \le C_7\|z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})},
\tag{5.22}
$$

where

$$
C_7 := \frac{1}{2\sqrt{2}}\left(\|v_0\|_{H^2(\mathbb{R})} + \|w_0\|_{H^2(\mathbb{R})}\right).
$$

Finally, we estimate I_4. From (5.13), (5.15), and [26, Theorem 8.5],

$$
\begin{aligned}
|I_4(t)| &\le \frac{|\mu-\lambda|^2}{2(\theta\mu+(1-\theta)\lambda)}\int_{\mathbb{R}} \left(\partial_x^3\omega_\theta\right)^2 dx\\
&\quad + \frac{1}{2(\theta\mu+(1-\theta)\lambda)}\int_{\mathbb{R}} z_\theta^2\left(\partial_x^2\omega_\theta\right)^2 dx\\
&\le \frac{|\mu-\lambda|^2}{2(\theta\mu+(1-\theta)\lambda)}\int_{\mathbb{R}} \left(\partial_x^3\omega_\theta\right)^2 dx\\
&\quad + \frac{\|z_\theta\|^2_{L^\infty}}{2(\theta\mu+(1-\theta)\lambda)}\int_{\mathbb{R}} \left(\partial_x^2\omega_\theta\right)^2 dx\\
&\le \frac{|\mu-\lambda|^2}{2(\theta\mu+(1-\theta)\lambda)}\int_{\mathbb{R}} \left(\partial_x^3\omega_\theta\right)^2 dx\\
&\quad + \frac{\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})}\|\omega_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})}}{4(\theta\mu+(1-\theta)\lambda)}\\
&\le |\mu-\lambda|^2 H_\theta(t) + C_{8,\theta}\|z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})},
\end{aligned}
\tag{5.23}
$$

where

$$
H_\theta(t) := \left(\|\partial_x^3 v_0\|_{L^2(\mathbb{R})} + \|\partial_x^3 w_0\|_{L^2(\mathbb{R})}\right)^2 \frac{e^{C_5 t}}{2(\theta\mu+(1-\theta)\lambda)}
$$

$$
+\frac{C_6(e^{C_5 t}-1)}{2(\theta\mu+(1-\theta)\lambda)C_5},
$$

$$
C_{8,\theta} := \frac{\big(\|v_0\|_{H^2(\mathbb{R})}+\|w_0\|_{H^2(\mathbb{R})}\big)^2}{4(\theta\mu+(1-\theta)\lambda)}.
$$

From (5.19), (5.20), (5.21), (5.23), and (5.22),

$$
\frac{d}{dt}\|z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})} \le \alpha_\theta \|z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})} + \beta_\theta(t)|\mu-\lambda|^2, \tag{5.24}
$$

where

$$
\alpha_\theta := 1+2\sqrt{2}\big(\|v_0\|_{H^2(\mathbb{R})}+\|w_0\|_{H^2(\mathbb{R})}\big)+2C_7+2C_{8,\theta},
$$

$$
\beta_\theta(t) := \big(\|v_0\|_{H^2(\mathbb{R})}+\|w_0\|_{H^2(\mathbb{R})}\big)^2 + H_\theta(t).
$$

Since

$$
0\le\alpha_\theta\le A, \qquad 0\le\beta_\theta(t)\le 2B_T^2, \qquad \text{for each } 0\le t\le T,\ 0\le\theta\le 1,
$$

from (5.24) and the Gronwall inequality

$$
\|z_\theta(t,\cdot)\|^2_{H^2(\mathbb{R})} \le e^{At}\|z_\theta(0,\cdot)\|^2_{H^2(\mathbb{R})} + 2B_T^2|\mu-\lambda|^2\frac{e^{At}-1}{A},
$$

for every $0\le t\le T$. □

We can now present the short proof of Theorem 5.1.

Proof of Theorem 5.1. The claim is a direct consequence of (5.16) and (5.18). □

6. The semigroup of solutions

In this section we prove Theorem 2.4 for $k=2$. We begin by proving that the family of viscous approximants is compact in the space $L^\infty_{\mathrm{loc}}([0,\infty);\mathcal{H}_{2,p})$, and the converging subsequence tends to a weak solution of (2.8). Moreover, we have that the weak limit of the vanishing viscosity approximants is unique and defines a semigroup of solutions in $\mathcal{H}_{2,p}$ satisfying the invariance properties (2.12) and (2.13). Finally, the weak limit of the vanishing viscosity approximants depends continuously on the initial condition, that is, the semigroup is strongly continuous in the sense of (2.11).

We start by establishing the compactness of the viscous approximants.

Lemma 6.1. *Let $2<p<\infty$. Assume that $u_0\in\mathcal{H}_{2,p}$. Then the family $\{u_\varepsilon\}_{\varepsilon>0}$ that solves* (2.14) *for $k=2$ is compact in $L^\infty_{\mathrm{loc}}([0,\infty);H^2(\mathbb{R}))$. Thus there exists a positive sequence $\{\varepsilon_h\}_{h\in\mathbb{N}}$ decreasing to* 0 *and a function $u\in L^\infty([0,\infty);H^2(\mathbb{R}))\cap H^1([0,T];H^1(\mathbb{R}))$, for each $T>0$, such that*

(*i*) *$u_{\varepsilon_h}\to u$ in $L^\infty([0,T];H^2(\mathbb{R}))$, for each $T>0$;*
(*ii*) *u is a weak solution of* (2.8) *for $k=2$.*

Before we prove this lemma, we need to establish some further properties. We begin with the following result on basic compactness.

Lemma 6.2. *Let $2<p<\infty$. Assume that $u_0\in\mathcal{H}_{2,p}$. Let u_ε, P_ε and $\widetilde{P}_\varepsilon$ be given by Lemma* 3.1*, equations* (4.1) *and* (4.25)*, respectively. There exists a positive sequence $\{\varepsilon_h\}_{h\in\mathbb{N}}$ decreasing to zero and three functions $u\in L^\infty([0,\infty);H^2(\mathbb{R}))\cap H^1([0,T];H^1(\mathbb{R}))$, which is a subset of $C([0,\infty);C^1(\mathbb{R}))$ for each $T>0$, $P\in L^\infty([0,\infty);W^{2,\infty}(\mathbb{R}))$ and $\widetilde{P}\in L^\infty([0,\infty);W^{1,\infty}(\mathbb{R}))$ such that*

$$
u_{\varepsilon_h} \rightharpoonup u \quad \text{weakly in } H^1([0,T];H^1(\mathbb{R})),\ \text{for each } T\ge 0; \tag{6.1}
$$

$$
u_{\varepsilon_h} \to u \quad \text{strongly in } L^\infty_{\mathrm{loc}}([0,\infty);H^1_{\mathrm{loc}}(\mathbb{R})); \tag{6.2}
$$

$$
P_{\varepsilon_h} \to P \quad \text{strongly in } L^p_{\mathrm{loc}}([0,\infty);W^{1,p}_{\mathrm{loc}}(\mathbb{R})),\ \text{for each } 1\le p<\infty; \tag{6.3}
$$

$$\widetilde{P}_{\varepsilon_h} \to \widetilde{P} \quad \textit{strongly in } L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}), \textit{ for each } 1 \le p < \infty. \tag{6.4}$$

Proof. Due to Lemmas 3.2, 4.1, 4.2, 4.3, and [28, Theorem 5], we can argue as in [4, Lemmas 5.2 and 5.3]. □

Denoting

$$q := \partial_x^2 u, \quad \text{in the weak sense,}$$

we infer from (4.24), (6.1), (6.2), and (6.4) that

$$\partial_t q + u\partial_x q + \widetilde{P} = 0, \tag{6.5}$$

holds in the sense of distributions in $[0,\infty)\times\mathbb{R}$.

Since in $\widetilde{P}$ we have the nonlinear term $(\partial_x^2 u)^2 = q^2$, we need to show that q_ε converges to q (strongly) in L^2. This convergence is needed if we want to send $\varepsilon \to 0$ in the viscous problem and recover the original problem.

Lemma 6.3. *Let $2 < p < \infty$. Assume that $u_0 \in \mathcal{H}_{2,p}$. Then there exists a map $\overline{q^2} \in L^\infty([0,\infty); L^r(\mathbb{R}))$, $1 \le r \le \frac{p}{2}$, such that for a subsequence we have*

$$q^2_{\varepsilon_h} \rightharpoonup \overline{q^2}, \qquad \textit{weakly in } L^\rho([0,T]; L^r(\mathbb{R})), \tag{6.6}$$

for each $T \ge 0$ and $1 < \rho < \infty$, $1 < r \le \frac{p}{2}$. Moreover,

$$q^2 \le \overline{q^2} \quad \textit{a.e. in } [0,\infty)\times\mathbb{R}, \tag{6.7}$$

and the following inequality holds

$$\partial_t \overline{q^2} - \partial_x u \overline{q^2} + \partial_x\big(u\overline{q^2}\big) + 2\widetilde{P}\,q \le 0, \tag{6.8}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$.

Proof. (6.6) follows from (2.9), (3.1) and Lemma 4.4.

The inequality (6.7) is a well-known consequence of Jensen's inequality.

Finally, we prove (6.8). Multiplying (4.24) by $2q_\varepsilon$ we get

$$\partial_t(q_\varepsilon^2) - \partial_x u_\varepsilon q_\varepsilon^2 + \partial_x\big(u_\varepsilon q_\varepsilon^2\big) + 2\widetilde{P}_\varepsilon q_\varepsilon = \varepsilon\partial_x^2\big(q_\varepsilon^2\big) - 2\varepsilon q_\varepsilon^2 \le \varepsilon\partial_x^2\big(q_\varepsilon^2\big),$$

hence (6.8) is consequence of (6.1), (6.2), (6.4) and (6.6). □

Arguing as in [5, Lemma 5.8], [29, Proposition 4.3], we get the following result.

Lemma 6.4. *Let $2 < p < \infty$. Assume that $u_0 \in \mathcal{H}_{2,p}$. The following identity holds*

$$\partial_t(q^2) - \partial_x u q^2 + \partial_x\big(uq^2\big) + 2\widetilde{P}q = 0, \tag{6.9}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$.

Proof of Lemma 6.1. We claim that

$$q_{\varepsilon_h} \to q \quad \text{strongly in } L^\infty([0,T]; L^2(\mathbb{R})), \text{ for each } T \ge 0. \tag{6.10}$$

Subtract (6.8) and (6.9)

$$\partial_t\big[\overline{q^2} - q^2\big] - \partial_x u\big[\overline{q^2} - q^2\big] + \partial_x\big[u(\overline{q^2} - q^2)\big] \le 0, \tag{6.11}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$.

Since (see, e.g., [5, Lemma 6.1])

$$\lim_{t\to 0}\int_{\mathbb{R}} q^2 dx = \lim_{t\to 0}\int_{\mathbb{R}} \overline{q^2} dx = \int_{\mathbb{R}} (\partial_x^2 u_0)^2 dx,$$

the claim is direct consequence of (6.11). □

We now turn to the question of uniqueness.

Lemma 6.5. *Let $2 < p < \infty$. Assume that $u_0 \in \mathcal{H}_{2,p}$. The limit of the family of the vanishing viscosity approximants is unique. As a consequence we have that*

$$u_\varepsilon \to u \quad \text{as } \varepsilon \to 0, \text{ strongly in } L^\infty([0,T];H^2(\mathbb{R})), \tag{6.12}$$

for each $T \geq 0$, where u is the function in Lemma 6.1.

Proof. Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$, $\{\mu_h\}_{h\in\mathbb{N}}$, ε_h, $\mu_h \to 0$, and u, $v \in L^\infty([0,T];H^2(\mathbb{R}))$, for each $T \geq 0$, be such that

$$u_{\varepsilon_h} \to u, \quad u_{\mu_h} \to v, \quad \text{strongly in } L^\infty([0,T];H^2(\mathbb{R})), \text{ for each } T \geq 0. \tag{6.13}$$

We have to prove that

$$u = v. \tag{6.14}$$

Let $0 < t < T$, it is not restrictive to assume that

$$\|u_{0,\varepsilon} - u_{0,\mu}\|_{H^2(\mathbb{R})} \leq |\varepsilon - \mu|, \qquad \varepsilon,\ \mu > 0. \tag{6.15}$$

Moreover, passing to subsequences, we can assume that

$$0 < \mu_n < \varepsilon_n < \mu_{n-1}, \qquad n \in \mathbb{N}. \tag{6.16}$$

Indeed, we can argue in the following way: we begin by considering two strictly decreasing subsequences $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$, $\{\mu_{n_k}\}_{k\in\mathbb{N}}$. Then we start by defining $\mu_{n_{k_0}} = \mu_{n_0}$, then we continue with $\varepsilon_{n_{k_1}}$ and $\mu_{n_{k_1}}$ in the following way:

$$\varepsilon_{n_{k_1}} = \max\left\{\varepsilon_{n_k} \mid \varepsilon_{n_k} < \mu_{n_{k_0}}\right\}, \qquad \mu_{n_{k_1}} = \max\left\{\mu_{n_k} \mid \mu_{n_k} < \varepsilon_{n_{k_1}}\right\}.$$

Arguing inductively in this way we find two subsequences $\{\varepsilon_{n_{k_h}}\}_{h\in\mathbb{N}}$, $\{\mu_{n_{k_h}}\}_{h\in\mathbb{N}}$ satisfying (6.16).

From Theorem 5.1 and (6.15), we have that

$$\begin{aligned}
&\|u_{\varepsilon_h}(t,\cdot\,) - u_{\mu_h}(t,\cdot\,)\|_{H^2(\mathbb{R})} \\
&\qquad \leq e^{A(T,\varepsilon_h,\mu_h)}\|u_{0,\varepsilon_h} - u_{0,\mu_h}\|_{H^2(\mathbb{R})} + B(T,\varepsilon_h,\mu_h,h)|\varepsilon_h - \mu_h| \\
&\qquad \leq \left(e^{A(T,\varepsilon_h,\mu_h)} + B(T,\varepsilon_h,\mu_h,h)\right)|\varepsilon_h - \mu_h|,
\end{aligned}$$

with

$$\begin{aligned}
&0 \leq A(T,\varepsilon_h,\mu_h) \leq \delta T + \frac{\delta T}{\min\{\varepsilon_h,\mu_h\}}, \\
&0 \leq B(T,\varepsilon_h,\mu_h,h) \leq \delta e^{\delta T} e^{(\delta T)/(\varepsilon_h+\mu_h)}\left(1 + \frac{m_h e^{\delta T}}{\sqrt{\min\{\varepsilon_h,\mu_h\}}} + \frac{e^{\delta T}}{\sqrt{\min\{\varepsilon_h,\mu_h\}}}\right),
\end{aligned}$$

where δ depends only on $\sup_k \|u_{0,k}\|_{H^2(\mathbb{R})}$ and

$$m_h := \sup_{k=0,\dots,h} \|\partial_x^3 u_{0,k}\|_{L^2(\mathbb{R})}.$$

Hence

$$\|u_{\varepsilon_h}(t,\cdot\,) - u_{\mu_h}(t,\cdot\,)\|_{H^2(\mathbb{R})} \leq c_1 \frac{e^{\delta T/(\varepsilon_h+\mu_h)} m_h}{\sqrt{\min\{\varepsilon_h,\mu_h\}}}, \tag{6.17}$$

for some constant $c_1 > 0$. Define

$$\varepsilon_{k,n} := \varepsilon_n - k\frac{e^{-\delta T/\varepsilon_n^2}\sqrt{\varepsilon_n}}{m_n}, \quad N_n := \left\lfloor (\varepsilon_n - \mu_n)\frac{e^{\delta T/\varepsilon_n^2} m_n}{\sqrt{\varepsilon_n}} \right\rfloor, \qquad k, n \in \mathbb{N},$$

where $\lfloor\cdot\rfloor$ denote the integer part. Observe that

$$\varepsilon_{N_n,n} \leq \mu_n \leq \varepsilon_{N_n-1,n}, \quad \mu_n - \varepsilon_{N_n,n} \leq \frac{e^{-\delta T/\varepsilon_n^2}\sqrt{\varepsilon_n}}{m_n}, \qquad n \in \mathbb{N}, \tag{6.18}$$

and

$$
\begin{aligned}
(6.19)\qquad \lim_n \|u_{\varepsilon_{k,n}}(t,\cdot)-u_{\varepsilon_n}(t,\cdot)\|_{H^2(\mathbb{R})}
&\le c_1 \lim_n \frac{e^{\delta T/(\varepsilon_n+\varepsilon_{k,n})}m_n}{\sqrt{\varepsilon_n}}|\varepsilon_{k,n}-\varepsilon_n|\\
&\le c_1 k \lim_n \frac{e^{\delta T/\varepsilon_n}m_n}{\sqrt{\varepsilon_n}}\frac{e^{-\delta T/\varepsilon_n^2}\sqrt{\varepsilon_n}}{m_n}=0,
\end{aligned}
$$

for each $k\in\mathbb{N}$, in other terms

$$
(6.20)\qquad u_{\varepsilon_{k,n}}\to u, \quad \text{strongly in } L^\infty([0,T];H^2(\mathbb{R})) \text{ as } n\to\infty, \quad \text{for each } k\in\mathbb{N}.
$$

Since, from (6.16),

$$
\varepsilon_{n+1}<\mu_n<\mu_n+\varepsilon_{N_n,n}, \qquad n\in\mathbb{N},
$$

employing (6.18), we have that

$$
\begin{aligned}
(6.21)\qquad \lim_n \|u_{\varepsilon_{N_n,n}}(t,\cdot)-u_{\mu_n}(t,\cdot)\|_{H^2(\mathbb{R})}
&\le c_1 \lim_n \frac{e^{\delta T/\varepsilon_{N_n,n}}m_{n+1}}{\sqrt{\varepsilon_{N_n,n}}}|\varepsilon_{N_n,n}-\mu_n|\\
&\le c_1 \lim_n \frac{e^{\delta T/\varepsilon_{n+1}}m_{n+1}}{\sqrt{\varepsilon_{n+1}}}\frac{e^{-\delta T/\varepsilon_n^2}\sqrt{\varepsilon_n}}{m_n}=0.
\end{aligned}
$$

Hence

$$
(6.22)\qquad u_{\varepsilon_{N_n,n}}\to v, \quad \text{strongly in } L^\infty([0,T];H^2(\mathbb{R})) \text{ as } n\to\infty.
$$

If $\{N_n\}_{n\in\mathbb{N}}$ is bounded, the claim is direct consequence of (6.20) and (6.22). So we consider the case

$$
(6.23)\qquad \lim_n N_n=\infty.
$$

As before, we define the sequences

$$
\mu_{h,n}:=\mu_n+h\frac{e^{-\delta T/\mu_n^2}\sqrt{\mu_n}}{m_n}, \quad M_n:=\left\lfloor(\varepsilon_n-\mu_n)\frac{e^{\delta T/\mu_n^2}m_n}{\sqrt{\mu_n}}\right\rfloor, \qquad h,n\in\mathbb{N}.
$$

Due to (6.16) and (6.23),

$$
(6.24)\qquad \lim_n M_n=\infty,
$$

and, arguing as for (6.20) and (6.22), we are able to prove that

$$
\begin{aligned}
(6.25)\qquad \lim_n \|u_{\mu_{h,n}}(t,\cdot)-u_{\mu_n}(t,\cdot)\|_{H^2(\mathbb{R})}
&=\lim_n \|u_{\mu_{h,n}}(t,\cdot)-v(t,\cdot)\|_{H^2(\mathbb{R})}=0, \quad h\in\mathbb{N},\\
(6.26)\qquad \lim_n \|u_{\mu_{M_n,n}}(t,\cdot)-u_{\varepsilon_n}(t,\cdot)\|_{H^2(\mathbb{R})}
&=\lim_n \|u_{\mu_{M_n,n}}(t,\cdot)-u(t,\cdot)\|_{H^2(\mathbb{R})}=0.
\end{aligned}
$$

Due to (6.23) and (6.24), we can choose two sequences $\{k_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}}$, such that

$$
(6.27)\qquad \mu_n\le\mu_{h_n,n},\varepsilon_{k_n,n}\le\varepsilon_n, \quad |\mu_{h_n,n}-\varepsilon_{k_n,n}|\le c_2\frac{e^{-\delta T/\mu_n^2}\sqrt{\mu_n}}{m_n}, \qquad n\in\mathbb{N},
$$

$$
(6.28)\qquad \lim_n h_n=\lim_n k_n=\infty,
$$

for some constant $c_2>0$. Observe that

$$
(6.29)\qquad \|u(t,\cdot)-v(t,\cdot)\|_{H^2(\mathbb{R})}\le\|u(t,\cdot)-u_{\mu_{h,n}}(t,\cdot)\|_{H^2(\mathbb{R})}
$$

$$+ \|u_{\mu_{h,n}}(t,\cdot) - u_{\varepsilon_{k,n}}(t,\cdot)\|_{H^2(\mathbb{R})}$$
$$+ \|u_{\varepsilon_{k,n}}(t,\cdot) - v(t,\cdot)\|_{H^2(\mathbb{R})}.$$

From (6.22) and (6.26), we have

$$\liminf_{h,n} \|u(t,\cdot) - u_{\mu_{h,n}}(t,\cdot)\|_{H^2(\mathbb{R})} \le \lim_n \|u(t,\cdot) - u_{\mu_{M_n,n}}(t,\cdot)\|_{H^2(\mathbb{R})} = 0, \tag{6.30}$$

$$\liminf_{k,n} \|u_{\varepsilon_{k,n}}(t,\cdot) - v(t,\cdot)\|_{H^2(\mathbb{R})} \le \lim_n \|u_{\varepsilon_{N_n,n}}(t,\cdot) - v(t,\cdot)\|_{H^2(\mathbb{R})} = 0, \tag{6.31}$$

respectively. Finally, from (6.17), (6.27), and (6.28),

$$\begin{aligned} \liminf_{h,k,n} &\|u_{\mu_{h,n}}(t,\cdot) - u_{\varepsilon_{k,n}}(t,\cdot)\|_{H^2(\mathbb{R})} \\ &\le \liminf_n \|u_{\mu_{h_n,n}}(t,\cdot) - u_{\varepsilon_{k_n,n}}(t,\cdot)\|_{H^2(\mathbb{R})} \\ &\le c_1c_2 \liminf_n \frac{e^{-\delta T/\mu_n^2}\sqrt{\mu_n}}{m_n} \frac{e^{\delta T/(\mu_{h_n,n}+\varepsilon_{k_n,n})} m_n}{\sqrt{\min\{\mu_{h_n,n}, \varepsilon_{k_n,n}\}}} \\ &\le c_1c_2 \lim_n \frac{e^{-\delta T/\mu_n^2}\sqrt{\mu_n}}{m_n} \frac{e^{\delta T/\mu_n} m_n}{\sqrt{\mu_n}} = 0. \end{aligned} \tag{6.32}$$

Clearly, (6.29), (6.30), (6.31), (6.32), imply $u = v$. □

Due to this lemma we can now define the semigroup

$$S\colon [0,T] \times \mathcal{H}_{2,p}(\mathbb{R}) \to L^\infty([0,T]; H^2(\mathbb{R})),$$

$u(t,x) = S_t(u_0)(x)$ is the limit of the vanishing viscosity approximants. Clearly S is a semigroup of solutions associated with the Cauchy problem (2.8). Moreover, the invariance properties (2.12) and (2.13) are direct consequences of Lemmas 4.4, 4.5, 6.1, and 6.5.

We now turn to the question of stability with respect to initial data.

Lemma 6.6. *Let $2 < p < \infty$, $\{u_{0,n}\}_{n\in\mathbb{N}} \subset \mathcal{H}_{2,p}$, $u_0 \in \mathcal{H}_{2,p}$ such that*

$$u_{0,n} \to u_0 \quad \textit{strongly in } H^2(\mathbb{R}). \tag{6.33}$$

There results

$$S(u_{0,n}) \to S(u_0) \ \textit{ strongly in } L^\infty([0,T]; H^2(\mathbb{R})), \tag{6.34}$$

for each $T > 0$.

Proof. Let $\varepsilon > 0$. Consider the families $\{u_{0,\varepsilon,n}\}_{\varepsilon>0,n\in\mathbb{N}}$, $\{u_{0,\varepsilon}\}_{\varepsilon>0} \subset H^3(\mathbb{R})$, such that

$$u_{0,\varepsilon,n} \to u_{0,n}, \quad u_{0,\varepsilon} \to u_0, \quad \text{in } H^2(\mathbb{R}) \text{ as } \varepsilon \to 0, \tag{6.35}$$

$$\|u_{0,\varepsilon,n} - u_{0,\varepsilon}\|_{H^2(\mathbb{R})} \le \|u_{0,n} - u_0\|_{H^2(\mathbb{R})}, \quad \varepsilon > 0,\ n \in \mathbb{N}. \tag{6.36}$$

Denote by S^ε the semigroup associated with the viscous problem (2.17). If $0 < t < T$, then

$$\begin{aligned} \|S_t(u_{0,n}) - S_t(u_0)\|_{H^2(\mathbb{R})} \le{}& \|S_t(u_{0,n}) - S_t^\varepsilon(u_{0,\varepsilon,n})\|_{H^2(\mathbb{R})} \\ &+ \|S_t^\varepsilon(u_{0,\varepsilon,n}) - S_t^\varepsilon(u_{0,\varepsilon})\|_{H^2(\mathbb{R})} \\ &+ \|S_t^\varepsilon(u_{0,\varepsilon}) - S_t(u_0)\|_{H^2(\mathbb{R})}, \end{aligned} \tag{6.37}$$

so that

$$\begin{aligned} 0 \le \liminf_n \|S_t(u_{0,n}) - S_t(u_0)\|_{H^2(\mathbb{R})} \le{}& \liminf_{\varepsilon,n} \|S_t(u_{0,n}) - S_t^\varepsilon(u_{0,\varepsilon,n})\|_{H^2(\mathbb{R})} \\ &+ \liminf_{\varepsilon,n} \|S_t^\varepsilon(u_{0,\varepsilon,n}) - S_t^\varepsilon(u_{0,\varepsilon})\|_{H^2(\mathbb{R})} \\ &+ \liminf_{\varepsilon} \|S_t^\varepsilon(u_{0,\varepsilon}) - S_t(u_0)\|_{H^2(\mathbb{R})}. \end{aligned} \tag{6.38}$$

From Lemma 6.1 and (6.35), we know that

$$\liminf_{\varepsilon,n} \|S_t(u_{0,n}) - S_t^\varepsilon(u_{0,\varepsilon,n})\|_{H^2(\mathbb{R})} = 0, \tag{6.39}$$

$$\liminf_{\varepsilon} \|S_t^\varepsilon(u_{0,\varepsilon}) - S_t(u_0)\|_{H^2(\mathbb{R})} = 0. \tag{6.40}$$

We claim that

$$\liminf_{\varepsilon,n} \|S_t^\varepsilon(u_{0,\varepsilon,n}) - S_t^\varepsilon(u_{0,\varepsilon})\|_{H^2(\mathbb{R})} = 0. \tag{6.41}$$

Using Theorem 5.1 and (6.36), we have that

$$\begin{aligned} \|S_t^\varepsilon(u_{0,\varepsilon,n}) - S_t^\varepsilon(u_{0,\varepsilon})\|_{H^2(\mathbb{R})} &\le e^{A(T,\varepsilon)} \|u_{0,\varepsilon,n} - u_{0,\varepsilon}\|_{H^2(\mathbb{R})} \\ &\le e^{A(T,\varepsilon)} \|u_{0,n} - u_0\|_{H^2(\mathbb{R})}, \end{aligned} \tag{6.42}$$

with

$$0 \le A(T,\varepsilon) \le \frac{\gamma T}{\varepsilon},$$

where γ depends only on $\sup_{\varepsilon,n} \|u_{0,\varepsilon,n}\|_{H^2(\mathbb{R})}$.

Define

$$\varepsilon_n := \left| \log(\|u_{0,n} - u_0\|_{H^2(\mathbb{R})}^{1/2}) \right|^{-1}.$$

Clearly

$$\liminf_{\varepsilon,n} \|S_t^\varepsilon(u_{0,n}) - S_t^\varepsilon(u_0)\|_{H^2(\mathbb{R})} \le \liminf_{n} \|S_t^{\varepsilon_n}(u_{0,n}) - S_t^\varepsilon(u_0)\|_{H^2(\mathbb{R})}, \tag{6.43}$$

and

$$\lim_{n} A(T,\varepsilon_n) \|u_{0,n} - u_0\|_{H^2(\mathbb{R})} = 0. \tag{6.44}$$

Then (6.41) is consequence of (6.42), (6.43), and (6.44). From (6.38), (6.39), (6.40), and (6.41), we get

$$\lim_{n} \|S_t(u_{0,n}) - S_t(u_0)\|_{H^2(\mathbb{R})} = 0.$$

□

7. Weak Equals Strong Uniqueness

In this section we prove Theorem 2.5.

The following lemma is needed.

Lemma 7.1. *Assume $k = 2$. Let u_1, u_2 be two weak solutions of the system* (2.10) *in the sense of Definition* 2.3. *If there exists a map $b \in L^1([0,T])$, $T > 0$, such that*

$$\left\|\partial_x^2 u_1(t,\cdot)\right\|_{L^\infty(\mathbb{R})}, \left\|\partial_x^2 u_2(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \le b(t), \qquad t \ge 0, \tag{7.1}$$

then,

$$\mathcal{L}(t) \le \mathcal{L}(0) + c\int_0^t (1 + b(s))\mathcal{L}(s)ds, \tag{7.2}$$

for each $t \ge 0$ and some constant $c > 0$, where

$$\begin{aligned} \mathcal{L}(t) = &\|w(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(t,\cdot)\|_{L^\infty(\mathbb{R})} \\ &+ \left\|(\partial_x^3 - \partial_x)A_2^{-1} e(t,\cdot)\right\|_{L^\infty(\mathbb{R})} + \left\|A_2^{-2} e(t,\cdot)\right\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Proof. Introduce the notation

$$w = u_1 - u_2, \quad v = u_1 + u_2, \quad e_i = u_i^2 + (\partial_x u_i)^2 + (\partial_x^2 u_i)^2, \quad e = \frac{1}{2}(e_1 - e_2).$$

We split the proof in six steps.

Step 1. We begin with some manipulation of the equations. By fixed $i \in \{1,2\}$, since

$$P_i = -\frac{1}{2}A_2^{-1}e_i + A_2^{-1}\left(\frac{3}{2}u_i^2 + (\partial_x u_i)^2\right) - \frac{3}{2}\partial_x^2 A_2^{-1}\left((\partial_x u_i)^2\right),$$

$$\partial_x^4 A_2^{-1} = \partial_x^2 A_2^{-1} - A_2^{-1} + 1,$$

we have that

$$\begin{aligned}
\partial_x P_i =& -\frac{1}{2}\partial_x A_2^{-1}e_i + F_i,\\
\partial_x^2 P_i =& -\frac{1}{2}\partial_x^2 A_2^{-1}e_i - \frac{3}{2}(\partial_x u_i)^2 + G_i,\\
\partial_x^3 P_i =& -\frac{1}{2}\partial_x^3 A_2^{-1}e_i - 3\partial_x u_i \partial_x^2 u_i + \partial_x G_i,
\end{aligned}$$

where

$$\begin{aligned}
F_i =& \partial_x A_2^{-1}\left(\frac{3}{2}u_i^2 + (\partial_x u_i)^2\right) - \frac{3}{2}\partial_x^3 A_2^{-1}\left((\partial_x u_i)^2\right),\\
G_i =& \partial_x^2 A_2^{-1}\left(\frac{3}{2}u_i^2 - \frac{1}{2}(\partial_x u_i)^2\right) + \frac{3}{2}A_2^{-1}\left((\partial_x u_i)^2\right).
\end{aligned}$$

Hence

$$\partial_t u_i + u_i\partial_x u_i - \frac{1}{2}\partial_x A_2^{-1}e_i + F_i = 0, \tag{7.3}$$

$$\partial_t\partial_x u_i + u_i\partial_x^2 u_i - \frac{1}{2}(\partial_x u_i)^2 - \frac{1}{2}\partial_x^2 A_2^{-1}e_i + G_i = 0, \tag{7.4}$$

$$\partial_t\partial_x^2 u_i + u_i\partial_x^3 u_i - \frac{1}{2}\partial_x^3 A_2^{-1}e_i + \partial_x G_i = 0. \tag{7.5}$$

In particular, from (7.3) and (7.4), we get the following equations for w, $\partial_x w$

$$\partial_t w + u_1\partial_x w + w\partial_x u_2 - \partial_x A_2^{-1}e + F_1 - F_2 = 0, \tag{7.6}$$

$$\partial_t\partial_x w + u_1\partial_x^2 w + w\partial_x^2 u_2 - \frac{1}{2}\partial_x w\partial_x v - \partial_x^2 A_2^{-1}e + G_1 - G_2 = 0. \tag{7.7}$$

Multiplying (7.3) by u_i, (7.4) by $\partial_x u_i$, (7.5) by $\partial_x^2 u_i$, adding the three equations, and observing

$$\begin{aligned}
&u_i\partial_x A_2^{-1}e_i + \partial_x u_i\partial_x^2 A_2^{-1}e_i + \partial_x^2 u_i\partial_x^3 A_2^{-1}e_i\\
&\qquad = - e_i\partial_x u_i + \partial_x\left(u_i A_2^{-1}e_i + \partial_x u_i\partial_x^3 A_2^{-1}e_i\right),
\end{aligned}$$

we get

$$\frac{1}{2}\partial_t e_i + \frac{1}{2}\partial_x(u_i e_i) - \frac{1}{2}(\partial_x u_i)^3 - \frac{1}{2}\partial_x\left(u_i A_2^{-1}e_i + \partial_x u_i\partial_x^3 A_2^{-1}e_i\right) + H_i = 0, \tag{7.8}$$

where

$$H_i = u_i F_i + \partial_x u_i G_i + \partial_x^2 u_i\partial_x G_i.$$

Finally, from (7.8) we get the following equation for e

$$\begin{aligned}
&\partial_t e + \partial_x(u_1 e) + \frac{1}{2}\partial_x(we_2) - \frac{1}{2}\left((\partial_x u_1)^3 - (\partial_x u_2)^3\right) + H_1 - H_2\\
&\quad - \partial_x\left(u_1 A_2^{-1}e + \frac{w}{2}A_2^{-1}e_2 + \partial_x u_1\partial_x^3 A_2^{-1}e + \frac{\partial_x w}{2}\partial_x^3 A_2^{-1}e_2\right) = 0.
\end{aligned} \tag{7.9}$$

Step 2. We estimate the L^∞-norm of w. Since $\partial_x u_1 \in L^\infty([0,\infty)\times\mathbb{R})$ applying [30, Lemma 2] to (7.6) we get

$$\begin{aligned}
\|w(t,\cdot)\|_{L^\infty(\mathbb{R})} \le& \|w(0,\cdot)\|_{L^\infty(\mathbb{R})}\\
&+ \int_0^t \left\|(w\partial_x u_2 - \partial_x A_2^{-1}e + F_1 - F_2)(s,\cdot)\right\|_{L^\infty(\mathbb{R})} ds,
\end{aligned} \tag{7.10}$$

for each $t > 0$. Using Definition 2.3 (*iv*) and the Sobolev embedding theorem [26, Theorem 8.5]

$$\|\partial_x u_i(s,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|u_i(s,\cdot)\|_{H^2(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|u_i(0,\cdot)\|_{H^2(\mathbb{R})}, \tag{7.11}$$

for any $s \geq 0$, $i = 1, 2$.

Moreover,

$$F_1 - F_2 = \partial_x A_2^{-1}\left(\frac{3}{2}wv + \partial_x w \partial_x v\right) - \frac{3}{2}\partial_x^3 A_2^{-1}\left(\partial_x w \partial_x v\right),$$

hence using the boundedness of the derivatives of the Green's function of A_2 and again Definition 2.3 (*iv*) and the Sobolev embedding theorem [26, Theorem 8.5]

$$\|(F_1 - F_2)(s,\cdot)\|_{L^\infty(\mathbb{R})} \leq c_1 \Big(\|w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s,\cdot)\|_{L^\infty(\mathbb{R})}\Big), \tag{7.12}$$

for each $s \geq 0$ and some constant $c_1 > 0$.

Therefore, by (7.10), (7.11), (7.12),

$$\begin{aligned}\|w(t,\cdot)\|_{L^\infty(\mathbb{R})} &\leq \|w(0,\cdot)\|_{L^\infty(\mathbb{R})} \\ &+ c_2 \int_0^t \Big(\|w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \left\|\partial_x A_2^{-1} e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\Big)\, ds,\end{aligned} \tag{7.13}$$

for each $t > 0$ and some constant $c_2 > 0$.

Step 3. We use the same argument for the estimate of the L^∞-norm of $\partial_x w$. The boundedness of $\partial_x u_1$, [30, Lemma 2] and (7.6) give

$$\begin{aligned}\|\partial_x w(t,\cdot)\|_{L^\infty(\mathbb{R})} &\leq \|\partial_x w(0,\cdot)\|_{L^\infty(\mathbb{R})} \\ &+ \int_0^t \left\|\left(w\partial_x^2 u_2 - \frac{1}{2}\partial_x w \partial_x v - \partial_x^2 A_2^{-1} e + G_1 - G_2\right)(s,\cdot)\right\|_{L^\infty(\mathbb{R})} ds,\end{aligned} \tag{7.14}$$

for each $t > 0$. (7.1) and (7.11) yield

$$\left\|\left(w\partial_x^2 u_2\right)(s,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq 2b(s)\, \|w(s,\cdot)\|_{L^\infty(\mathbb{R})}, \tag{7.15}$$

$$\|\left(\partial_x w \partial_x v\right)(s,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|v(0,\cdot)\|_{H^2(\mathbb{R})} \|\partial_x w(s,\cdot)\|_{L^\infty(\mathbb{R})}, \tag{7.16}$$

for each $s \geq 0$. Finally

$$G_1 - G_2 = \partial_x^2 A_2^{-1}\left(\frac{3}{2}wv + \frac{5}{2}\partial_x w \partial_x v\right) - \frac{3}{2}\partial_x^2 A_2^{-1}\left(\partial_x w \partial_x v\right),$$

hence using the boundedness of the derivatives of the Green's function of A_2 and again Definition 2.3 (*iv*) and the Sobolev embedding theorem [26, Theorem 8.5]

$$\|(G_1 - G_2)(s,\cdot)\|_{L^\infty(\mathbb{R})} \leq c_3 \Big(\|w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s,\cdot)\|_{L^\infty(\mathbb{R})}\Big), \tag{7.17}$$

for each $s \geq 0$ and some constant $c_3 > 0$.

Therefore, by (7.14), (7.15), (7.16), (7.17)

$$\begin{aligned}\|\partial_x w(t,\cdot)\|_{L^\infty(\mathbb{R})} &\leq \|\partial_x w(0,\cdot)\|_{L^\infty(\mathbb{R})} + c_4 \int_0^t (1 + b(s)) \\ &\times \Big(\|w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \left\|\partial_x^2 A_2^{-1} e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\Big)\, ds,\end{aligned} \tag{7.18}$$

for each $t > 0$ and some constant $c_4 > 0$.

Step 4. Introduce the operator

$$\Lambda = (\partial_x^3 - \partial_x) A_2^{-1}$$

and observe that

$$\partial_x \Lambda = (\partial_x^4 - \partial_x^2) A_2^{-1} = 1 - A_2^{-1}. \tag{7.19}$$

Applying Λ to (7.9)

$$\begin{aligned}(7.20)\quad &\partial_t \Lambda e + \partial_x \Lambda(u_1 e) + \frac{1}{2}\partial_x \Lambda(w e_2) - \frac{1}{2}\Lambda\left((\partial_x u_1)^3 - (\partial_x u_2)^3\right) + \Lambda(H_1 - H_2)\\ &- \partial_x \Lambda\left(u_1 A_2^{-1} e + \frac{w}{2} A_2^{-1} e_2 + \partial_x u_1 \partial_x^3 A_2^{-1} e + \frac{\partial_x w}{2}\partial_x^3 A_2^{-1} e_2\right) = 0.\end{aligned}$$

Due to (7.19)

$$(7.21)\quad \partial_x \Lambda(u_1 e) = u_1 e - A_2^{-1}(u_1 e) = u_1 \partial_x \Lambda e + u_1 A_2^{-1} e - A_2^{-1}(u_1 e),$$

$$(7.22)\quad \frac{1}{2}\partial_x \Lambda(w e_2) = \frac{1}{2} w e_2 - \frac{1}{2} A_2^{-1}(w e_2),$$

$$\begin{aligned}(7.23)\quad &- \partial_x \Lambda\left(u_1 A_2^{-1} e + \frac{w}{2} A_2^{-1} e_2 + \partial_x u_1 \partial_x^3 A_2^{-1} e + \frac{\partial_x w}{2}\partial_x^3 A_2^{-1} e_2\right)\\ &= -u_1 A_2^{-1} e - \frac{w}{2} A_2^{-1} e_2 - \partial_x u_1 \Lambda e - \partial_x u_1 \partial_x A_2^{-1} e - \frac{\partial_x w}{2}\partial_x^3 A_2^{-1} e_2\\ &+ A_2^{-1}\left(u_1 A_2^{-1} e + \frac{w}{2} A_2^{-1} e_2 + \partial_x u_1 \Lambda e + \partial_x u_1 \partial_x A_2^{-1} e + \frac{\partial_x w}{2}\partial_x^3 A_2^{-1} e_2\right).\end{aligned}$$

Using (7.21), (7.22), (7.23) in (7.20),

$$(7.24)\qquad \partial_t \Lambda e + u_1 \partial_x \Lambda e + h = 0,$$

where

$$\begin{aligned}h &= u_1 A_2^{-1} e - A_2^{-1}(u_1 e) + \frac{1}{2} w e_2 - \frac{1}{2} A_2^{-1}(w e_2) + \Lambda(H_1 - H_2)\\ &- u_1 A_2^{-1} e - \frac{w}{2} A_2^{-1} e_2 - \partial_x u_1 \Lambda e - \partial_x u_1 \partial_x A_2^{-1} e - \frac{\partial_x w}{2}\partial_x^3 A_2^{-1} e_2\\ &+ A_2^{-1}\left(u_1 A_2^{-1} e + \frac{w}{2} A_2^{-1} e_2 + \partial_x u_1 \Lambda e + \partial_x u_1 \partial_x A_2^{-1} e + \frac{\partial_x w}{2}\partial_x^3 A_2^{-1} e_2\right).\end{aligned}$$

Due to the boundedness of the derivatives of the Green's function of A_2 and again Definition 2.3 (*iv*) and the Sobolev embedding theorem [26, Theorem 8.5]

$$\begin{aligned}\|h(s,\cdot)\|_{L^\infty(\mathbb{R})} \le c_5 \Big(&\|w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s,\cdot)\|_{L^\infty(\mathbb{R})}\\ &+ \|\Lambda e(s,\cdot)\|_{L^\infty(\mathbb{R})} + \left\|\partial_x A_2^{-1} e(s,\cdot)\right\|_{L^\infty(\mathbb{R})} + \left\|A_2^{-1} e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\Big),\end{aligned}$$

for each $s \ge 0$ and some constant $c_5 > 0$. Hence, [30, Lemma 2] and (7.24) give

$$\begin{aligned}(7.25)\quad \|\Lambda e(t,\cdot)\|_{L^\infty(\mathbb{R})} &\le \|\Lambda e(0,\cdot)\|_{L^\infty(\mathbb{R})}\\ &+ c_5 \int_0^t \Big(\|w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s,\cdot)\|_{L^\infty(\mathbb{R})}\\ &+ \|\Lambda e(s,\cdot)\|_{L^\infty(\mathbb{R})} + \left\|\partial_x A_2^{-1} e(s,\cdot)\right\|_{L^\infty(\mathbb{R})} + \left\|A_2^{-1} e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\Big)\, ds,\end{aligned}$$

for each $s \ge 0$.

Step 5. Applying A_2^{-2} to (7.9)

$$\partial_t A_2^{-2} e + \partial_x A_2^{-2}(u_1 e) + k = 0,$$

where

$$\begin{aligned}k &= \frac{1}{2}\partial_x A_2^{-2}(w e_2) - \frac{1}{2} A_2^{-2}\left((\partial_x u_1)^3 - (\partial_x u_2)^3\right) + A_2^{-2}(H_1 - H_2)\\ &- \partial_x A_2^{-2}\left(u_1 A_2^{-1} e + \frac{w}{2} A_2^{-1} e_2 + \partial_x u_1 \partial_x^3 A_2^{-1} e + \frac{\partial_x w}{2}\partial_x^3 A_2^{-1} e_2\right).\end{aligned}$$

Due to the boundedness of the derivatives of the Green's function of A_2 and again Definition 2.3 (*iv*) and the Sobolev embedding theorem [26, Theorem 8.5]

$$\begin{aligned}\|k(s,\cdot)\|_{L^\infty(\mathbb{R})} \le c_6 \Big(&\|w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s,\cdot)\|_{L^\infty(\mathbb{R})}\\ &+ \left\|A_2^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})} + \left\|\partial_x A_2^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\Big),\end{aligned}$$

for each $s \ge 0$ and some constant $c_6 > 0$. Therefore integrating (7.26) on $(0,t)$ we have that

(7.26)
$$\begin{aligned}\left\|A_2^{-2}e(t,\cdot)\right\|_{L^\infty(\mathbb{R})} &\le \left\|A_2^{-2}e(0,\cdot)\right\|_{L^\infty(\mathbb{R})}\\ &\quad + \int_0^t \left(\left\|\partial_x A_2^{-2}(u_1 e)(s,\cdot)\right\|_{L^\infty(\mathbb{R})} + \|k(s,\cdot)\|_{L^\infty(\mathbb{R})}\right) ds\\ &\le \left\|A_2^{-2}e(0,\cdot)\right\|_{L^\infty(\mathbb{R})}\\ &\quad + c_7 \int_0^t \Big(\|w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s,\cdot)\|_{L^\infty(\mathbb{R})}\\ &\qquad + \left\|A_2^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})} + \left\|\partial_x A_2^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\Big) ds,\end{aligned}$$

for each $s \ge 0$ and some constant $c_7 > 0$.

Step 6. Adding together (7.13), (7.18), (7.25), (7.26)

(7.27)
$$\begin{aligned}&\|w(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|\Lambda e(t,\cdot)\| + \left\|A_2^{-2}e(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\\ &\le \|w(0,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(0,\cdot)\|_{L^\infty(\mathbb{R})} + \|\Lambda e(0,\cdot)\| + \left\|A_2^{-2}e(0,\cdot)\right\|_{L^\infty(\mathbb{R})}\\ &\quad + c_8 \int_0^t (1 + b(s))\\ &\quad \times \Big(\|w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s,\cdot)\|_{L^\infty(\mathbb{R})} + \|\Lambda e(s,\cdot)\|_{L^\infty(\mathbb{R})}\\ &\quad + \left\|\partial_x A_2^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})} + \left\|\partial_x^2 A_2^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})} + \left\|A_2^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\Big) ds,\end{aligned}$$

for each $s \ge 0$ and some constant $c_8 > 0$.

From (7.19)

$$A_2^{-1}e = \partial_x \Lambda A_2^{-1}e + A_2^{-2}e = \partial_x A_2^{-1}(\Lambda e) + A_2^{-2}e,$$

so

$$\left\|A_2^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})} \le c_9 \left(\|\Lambda e(s,\cdot)\|_{L^\infty(\mathbb{R})} + \left\|A_2^{-2}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\right), \tag{7.28}$$

for each $s \ge 0$ and some constant $c_9 > 0$. By fixed $j = 1, 2$ again by (7.19)

$$\partial_x^j A_2^{-1}e = \partial_x^{j+1}\Lambda A_2^{-1}e + \partial_x^j A_2^{-2}e = \partial_x^{j+1}A_2^{-1}(\Lambda e) + \partial_x^j A_2^{-1}(A_2^{-1}e),$$

using now (7.27) we have that

$$\begin{aligned}\left\|\partial_x^j A_2^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})} &\le c_{10}\left(\|\Lambda e(s,\cdot)\|_{L^\infty(\mathbb{R})} + \left\|A_2^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\right)\\ &\le c_{11}\left(\|\Lambda e(s,\cdot)\|_{L^\infty(\mathbb{R})} + \left\|A_2^{-2}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\right)\end{aligned} \tag{7.29}$$

for each $s \ge 0$ and some constant c_{10}, $c_{11} > 0$.

Finally, (7.27), (7.28), (7.29) imply (7.2). □

Proof of Theorem 2.5. Assume that there exist two weak solutions u_1, u_2 of the Cauchy problem (2.8) satisfying (7.1). The Gronwall lemma and (7.2) imply $\mathcal{L} = 0$ that means $u_1 = u_2$. □

Appendix A. Consistency of the weak formulation

In this section we prove the consistency of Definition 2.3 for a general k.

Lemma A.1. *Let $f \in C^\infty([0,\infty)\times\mathbb{R})$. Then*

$$f \in L^\infty([0,\infty); H^k(\mathbb{R})) \text{ implies } \int_{[0,\infty)\times\mathbb{R}} \mathcal{F}_k(f)\varphi dt dx < \infty, \tag{A.1}$$

for each $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$ and $k \geq 2$.

The following lemma is needed.

Lemma A.2. *Let $f \in C_c^\infty(\mathbb{R})$. The following identity holds*

$$\begin{aligned}\int_{-\infty}^{x} \big(2\partial_x f \partial_x^{2j} f + f\partial_x^{2j+1} f\big) d\xi \\ = f\partial_x^{2j} f + \sum_{i=1}^{j-1} (-1)^{i+1}\partial_x^i f \partial_x^{2j-i} f - \frac{(-1)^j}{2}\big(\partial_x^j f\big)^2,\end{aligned} \tag{A.2}$$

for each $x \in \mathbb{R}$ and $j \geq 2$.

Proof. Fix $f \in C_c^\infty(\mathbb{R})$, $x \in \mathbb{R}$ and $j \geq 2$. Integrating by parts we get

$$\begin{aligned}\int_{-\infty}^{x} \big(2\partial_x f\partial_x^{2j} f + f\partial_x^{2j+1} f\big)d\xi &= f\partial_x^{2j} f + \int_{-\infty}^{x} \partial_x f \partial_x^{2j} f d\xi \\ &= f\partial_x^{2j} f + \partial_x f \partial_x^{2j-1} f - \int_{-\infty}^{x} \partial_x^2 f \partial_x^{2j-1} f d\xi \\ &= f\partial_x^{2j} f + \partial_x f\partial_x^{2j-1} f - \partial_x^2 f \partial_x^{2j-2} f \\ &\quad + \int_{-\infty}^{x} \partial_x^3 f \partial_x^{2j-2} f d\xi \\ &= f\partial_x^{2j} f + \sum_{i=1}^{j-1}(-1)^{i+1}\partial_x^i f\partial_x^{2j-i} f \\ &\quad - (-1)^j \int_{-\infty}^{x} \partial_x^j f \partial_x^{j+1} f d\xi \\ &= f\partial_x^{2j} f + \sum_{i=1}^{j-1}(-1)^{i+1}\partial_x^i f\partial_x^{2j-i} f - \frac{(-1)^j}{2}\big(\partial_x^j f\big)^2.\end{aligned}$$

□

Proof of Lemma A.1. Let $f \in L^\infty([0,\infty); H^k(\mathbb{R}))$, $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$ and $k \geq 2$. From (A.2) we find, integrating by parts, that

$$\begin{aligned}\int_{[0,\infty)\times\mathbb{R}} \mathcal{F}_k(f)\varphi dt dx &= -\int_{[0,\infty)\times\mathbb{R}} \int_{-\infty}^{x} C_k(f)(t,y)\varphi(t,x) dt dx dy \\ &= \int_{[0,\infty)\times\mathbb{R}} \int_{-\infty}^{x} \big[A_k\big(f\partial_x f\big) - fA_k\big(\partial_x f\big) \\ &\qquad - 2\partial_x f A_k(f)\big](t,y)\varphi(t,x) dt dx dy \\ &= \sum_{j=1}^{k}(-1)^j \int_{[0,\infty)\times\mathbb{R}} \partial_x^{2j-1}\big(f\partial_x f\big)\varphi dt dx \\ &\quad + \int_{[0,\infty)\times\mathbb{R}} \int_{-\infty}^{x} \big[f\partial_x f\big](t,y)\varphi(t,x) dt dx dy\end{aligned} \tag{A.3}$$

$$
\begin{aligned}
&-\sum_{j=0}^{k}(-1)^j\int_{[0,\infty)\times\mathbb{R}}\int_{-\infty}^{x}\big[f\partial_x^{2j}\big(\partial_x f\big)\\
&\qquad+2\partial_x f\partial_x^{2j}(f)\big](t,y)\varphi(t,x)dtdxdy\\
&=J_1+J_2+J_3+J_4+J_5,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &:= \sum_{j=1}^{k}(-1)^j\int_{[0,\infty)\times\mathbb{R}}\partial_x^{2j-1}\big(f\partial_x f\big)\varphi dtdx,\\
J_2 &:= -\sum_{j=2}^{k}(-1)^j\int_{[0,\infty)\times\mathbb{R}}f\partial_x^{2j}f\varphi dtdx,\\
J_3 &:= -\sum_{j=2}^{k}\sum_{i=1}^{j-1}(-1)^{j+i+1}\int_{[0,\infty)\times\mathbb{R}}\partial_x^i f\partial_x^{2j-i}f\varphi dtdx,\\
J_4 &:= \frac{1}{2}\sum_{j=2}^{k}\int_{[0,\infty)\times\mathbb{R}}\big(\partial_x^j f\big)^2\varphi dtdx,\\
J_5 &:= \int_{[0,\infty)\times\mathbb{R}}\int_{-\infty}^{x}\big[f\partial_x f\big](t,y)\varphi(t,x)dtdxdy\\
&\quad-\sum_{j=0}^{1}(-1)^j\int_{[0,\infty)\times\mathbb{R}}\int_{-\infty}^{x}\big[f\partial_x^{2j}\big(\partial_x f\big)+2\partial_x f\partial_x^{2j}(f)\big](t,y)\varphi(t,x)dtdxdy.
\end{aligned}
$$

Observe that, employing integration by parts,

$$
\begin{aligned}
J_1 &= \sum_{j=1}^{k}\int_{[0,\infty)\times\mathbb{R}}f\partial_x f\partial_x^{2j-1}\varphi dtdx\\
&= -\sum_{j=1}^{k}\int_{[0,\infty)\times\mathbb{R}}\frac{f^2}{2}\partial_x^{2j}\varphi dtdx\\
&\le \frac{k}{2}\|f\|^2_{L^\infty([0,\infty);L^2(\mathbb{R}))}\|\varphi\|_{L^1([0,\infty);W^{2k,\infty}(\mathbb{R}))},
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
J_2 &= \sum_{j=2}^{k}\int_{[0,\infty)\times\mathbb{R}}\partial_x^j f\partial_x^j(f\varphi)dtdx\\
&\le c_1\|f\|^2_{L^\infty([0,\infty);H^k(\mathbb{R}))}\|\varphi\|_{L^1([0,\infty);W^{k,\infty}(\mathbb{R}))},
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
J_3 &= \sum_{j=2}^{k}\sum_{i=1}^{j-1}\int_{[0,\infty)\times\mathbb{R}}\partial_x^j f\partial_x^{j-i}(f\varphi)dtdx\\
&\le c_2\sum_{j=2}^{k}\|f\|^2_{L^\infty([0,\infty);H^j(\mathbb{R}))}\|\varphi\|_{L^1([0,\infty);W^{j,\infty}(\mathbb{R}))}\\
&\le c_3\|f\|^2_{L^\infty([0,\infty);H^k(\mathbb{R}))}\|\varphi\|_{L^1([0,\infty);W^{k,\infty}(\mathbb{R}))},
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
J_4 &\le \frac{1}{2}\sum_{j=2}^{k}(-1)^j\|f\|^2_{L^\infty([0,\infty);H^j(\mathbb{R}))}\|\varphi\|_{L^1([0,\infty);L^\infty(\mathbb{R}))}\\
&\le c_4\|f\|^2_{L^\infty([0,\infty);H^k(\mathbb{R}))}\|\varphi\|_{L^1([0,\infty);L^\infty(\mathbb{R}))},
\end{aligned}
\tag{A.7}
$$

$$
J_5 = \int_{[0,\infty)\times\mathbb{R}}\left[\frac{(\partial_x f)^2}{2}+f\partial_x^2 f-\frac{f^2}{2}\right]\varphi dtdx
\tag{A.8}
$$

$$\leq c_5 \|f\|^2_{L^\infty([0,\infty);H^2(\mathbb{R}))} \|\varphi\|_{L^1([0,\infty);L^\infty(\mathbb{R}))},$$

for some constants c_1, c_2, c_3, c_4, $c_5 > 0$ depending only on k. Clearly, (A.3), (A.4), (A.5), (A.6), (A.7), and (A.8) imply (A.2). □

Appendix B. The general case $k > 2$

In this appendix we show that the ideas used in the previous sections can be applied also in the general case. More precisely, we assume

$$k > 2.$$

Due to the boundedness of the family $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^\infty([0,\infty);H^k(\mathbb{R}))$ (see Lemma 3.2) as in Lemma 6.1, we have to prove compactness of the family $\{\partial_x^k u_\varepsilon\}_{\varepsilon>0}$ in $L^\infty([0,\infty);L^2(\mathbb{R}))$. To this end we derive an equation for $\partial_x^k u_\varepsilon$. From (2.14) we infer

$$\partial_t \partial_x^k u_\varepsilon + \partial_x^k \big(u_\varepsilon \partial_x u_\varepsilon\big) + \partial_x^{k+1} P_\varepsilon = \varepsilon \partial_x^{k+2} u_\varepsilon. \tag{B.1}$$

Observe that

$$\begin{aligned}
\partial_x^k \big(u_\varepsilon \partial_x u_\varepsilon\big) &= \sum_{j=0}^{k} \binom{k}{j} \partial_x^{j+1} u_\varepsilon \partial_x^{k-j} u_\varepsilon \\
&= u_\varepsilon \partial_x^{k+1} u_\varepsilon + (k+1)\partial_x u_\varepsilon \partial_x^{k-1} u_\varepsilon + U_\varepsilon,
\end{aligned} \tag{B.2}$$

where

$$U_\varepsilon := \sum_{j=1}^{k-2} \binom{k}{j} \partial_x^{j+1} u_\varepsilon \partial_x^{k-j} u_\varepsilon.$$

Since we have only derivatives in U_ε of order less than $k-2$, due to (3.1), it is uniformly bounded in $L^\infty([0,\infty);L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

In the following we analyze the nonlocal term $\partial_x^{k+1} P_\varepsilon$, employing Remark 2.1, and find

$$\begin{aligned}
\partial_x^{k+1} P_\varepsilon(t,x) &= \int_{\mathbb{R}} \frac{d^{k+1} G_k}{dx^{k+1}}(x-y)\mathcal{F}_k(t,y)dy \\
&= \int_{\mathbb{R}} \frac{d^k G_k}{dx^k}(x-y) C_k(t,y)dy = \big(P_{1,\varepsilon} + P_{2,\varepsilon} + P_{3,\varepsilon}\big)(t,x),
\end{aligned} \tag{B.3}$$

where

$$\begin{aligned}
P_{1,\varepsilon}(t,x) &:= -\int_{\mathbb{R}} \frac{d^k G_k}{dx^k}(x-y)[u_\varepsilon A_k(\partial_x u_\varepsilon)](t,y)dy, \\
P_{2,\varepsilon}(t,x) &:= \int_{\mathbb{R}} \frac{d^k G_k}{dx^k}(x-y) A_k(u_\varepsilon \partial_x u_\varepsilon)(t,y)dy, \\
P_{3,\varepsilon}(t,x) &:= -2\int_{\mathbb{R}} \frac{d^k G_k}{dx^k}(x-y)[\partial_x u_\varepsilon A_k(u_\varepsilon)](t,y)dy.
\end{aligned}$$

Observe that, integrating by parts and using the fact that G_k is the Green's function of A_k, we get

$$\begin{aligned}
P_{1,\varepsilon}(t,x) &= \sum_{j=0}^{k} (-1)^{j+1} \int_{\mathbb{R}} \frac{d^k G_k}{dx^k} u_\varepsilon \partial_x^{2j+1} u_\varepsilon dy \\
&= -\sum_{j=0}^{k} \int_{\mathbb{R}} \partial_x^j \Big(\frac{d^k G_k}{dx^k} u_\varepsilon\Big) \partial_x^{j+1} u_\varepsilon dy \qquad \text{(B.4)}\\
&= -\sum_{j=0}^{k-1} \int_{\mathbb{R}} \partial_x^j \Big(\frac{d^k G_k}{dx^k} u_\varepsilon\Big) \partial_x^{j+1} u_\varepsilon dy
\end{aligned}$$

$$
\begin{aligned}
&-(-1)^k u_\varepsilon \partial_x^{k+1} u_\varepsilon + \sum_{j=0}^{k-1} (-1)^{k+j} \int_{\mathbb{R}} \frac{d^{2j} G_k}{dx^{2j}} u_\varepsilon \partial_x^{k+1} u_\varepsilon dy \\
&- \sum_{j=0}^{k-1} \binom{k}{j} \int_{\mathbb{R}} \frac{d^{k+j} G_k}{dx^{k+j}} \partial_x^{k-j} u_\varepsilon \partial_x^{k+1} u_\varepsilon dy \\
= &- \sum_{j=0}^{k-1} \int_{\mathbb{R}} \partial_x^j \Big(\frac{d^k G_k}{dx^k} u_\varepsilon \Big) \partial_x^{j+1} u_\varepsilon dy \\
&-(-1)^k u_\varepsilon \partial_x^{k+1} u_\varepsilon - \sum_{j=0}^{k-1} (-1)^{k+j} \int_{\mathbb{R}} \partial_x \Big(\frac{d^{2j} G_k}{dx^{2j}} u_\varepsilon \Big) \partial_x^k u_\varepsilon dy \\
&- \sum_{j=0}^{k-2} \binom{k}{j} \int_{\mathbb{R}} \frac{d^{k+j+1} G_k}{dx^{k+j+1}} \partial_x^{k-j} u_\varepsilon \partial_x^k u_\varepsilon dy \\
&- \sum_{j=2}^{k-1} \binom{k}{j} \int_{\mathbb{R}} \frac{d^{k+j} G_k}{dx^{k+j}} \partial_x^{k-j+1} u_\varepsilon \partial_x^k u_\varepsilon dy + \frac{1}{2} \int_{\mathbb{R}} \frac{d^{k+2} G_k}{dx^{k+2}} \big(\partial_x^k u_\varepsilon \big)^2 dy \\
&+ (-1)^k k \partial_x u_\varepsilon \partial_x^k u_\varepsilon - \sum_{j=0}^{k-1} (-1)^{k+j} \int_{\mathbb{R}} \frac{d^{2j} G_k}{dx^{2j}} \partial_x u_\varepsilon \partial_x^k u_\varepsilon dy \\
&- \int_{\mathbb{R}} \frac{d^{2k-1} G_k}{dx^{2k-1}} \partial_x^{k+1} u_\varepsilon \partial_x^k u_\varepsilon dy \\
= &-(-1)^k u_\varepsilon \partial_x^{k+1} u_\varepsilon + (-1)^k k \partial_x u_\varepsilon \partial_x^k u_\varepsilon + \widetilde{P}_{1,\varepsilon},
\end{aligned}
$$

where $\widetilde{P}_{1,\varepsilon}$ is uniformly bounded in $L^\infty([0,\infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

Concerning the second term we have

$$
\begin{aligned}
P_{2,\varepsilon}(t,x) &= \sum_{j=0}^{k} (-1)^j \int_{\mathbb{R}} \frac{d^k G_k}{dx^k} \partial_x^{2j} (u_\varepsilon \partial_x u_\varepsilon) dy \\
&= \sum_{j=0}^{k-1} \int_{\mathbb{R}} \frac{d^{k+j} G_k}{dx^{k+j}} \partial_x^j (u_\varepsilon \partial_x u_\varepsilon) dy \\
&\quad + (-1)^k \partial_x^k (u_\varepsilon \partial_x u_\varepsilon) - \sum_{j=0}^{k-1} (-1)^{j+k} \int_{\mathbb{R}} \frac{d^{2j} G_k}{dx^{2j}} \partial_x^k (u_\varepsilon \partial_x u_\varepsilon) dy \\
&= \sum_{j=0}^{k-1} \int_{\mathbb{R}} \frac{d^{k+j} G_k}{dx^{k+j}} \partial_x^j (u_\varepsilon \partial_x u_\varepsilon) dy \\
&\quad + (-1)^k \partial_x^k (u_\varepsilon \partial_x u_\varepsilon) + \sum_{j=0}^{k-1} (-1)^{j+k} \int_{\mathbb{R}} \frac{d^{2j+1} G_k}{dx^{2j+1}} \partial_x^{k-1} (u_\varepsilon \partial_x u_\varepsilon) dy \\
&= (-1)^k u_\varepsilon \partial_x^{k+1} u_\varepsilon + (-1)^k (k+1) \partial_x u_\varepsilon \partial_x^k u_\varepsilon + \widetilde{P}_{2,\varepsilon},
\end{aligned}
\tag{B.5}
$$

where $\widetilde{P}_{2,\varepsilon}$ is uniformly bounded in $L^\infty([0,\infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

For the third term we have

(B.6)

$$
P_{3,\varepsilon}(t,x) = -2 \sum_{j=0}^{k} (-1)^j \int_{\mathbb{R}} \frac{d^k G_k}{dx^k} \partial_x u_\varepsilon \partial_x^{2j} u_\varepsilon dy
$$

$$
\begin{aligned}
&= -2\sum_{j=0}^{k-1}\int_{\mathbb{R}} \partial_x^j\Big(\frac{d^k G_k}{dx^k}\partial_x u_\varepsilon\Big)\partial_x^j u_\varepsilon dy \\
&\quad - 2(-1)^k \partial_x u_\varepsilon \partial_x^k u_\varepsilon + 2\sum_{j=0}^{k-1}(-1)^{k+j}\int_{\mathbb{R}} \frac{d^{2j}G_k}{dx^{2j}}\partial_x u_\varepsilon \partial_x^k u_\varepsilon dy \\
&\quad - 2\sum_{j=1}^{k-1}\binom{k}{j}\int_{\mathbb{R}} \frac{d^{k+j}G_k}{dx^{k+j}}\partial_x^{k+1-j}u_\varepsilon \partial_x^k u_\varepsilon dy + \int_{\mathbb{R}} \frac{d^{k+1}G_k}{dx^{k+1}}\big(\partial_x^k u_\varepsilon\big)^2 dy \\
&= -2(-1)^k \partial_x u_\varepsilon \partial_x^k u_\varepsilon + \widetilde{P}_{3,\varepsilon},
\end{aligned}
$$

where $\widetilde{P}_{3,\varepsilon}$ is uniformly bounded in $L^\infty([0,\infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

Hence, from (B.3), (B.4), (B.5), and (B.6) (as in (4.2)),

$$
\partial_x^k P_\varepsilon = (-1)^k(2k-1)\partial_x u_\varepsilon \partial_x^k u_\varepsilon + \widetilde{P}_\varepsilon, \tag{B.7}
$$

where $\widetilde{P}_\varepsilon$ is uniformly bounded in $L^\infty([0,\infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

Moreover, denoting

$$
q_\varepsilon := \partial_x^k u_\varepsilon
$$

from (B.1), (B.2), and (B.7), we get (as in (4.24))

$$
\partial_t q_\varepsilon + u_\varepsilon \partial_x q_\varepsilon + \lambda_\varepsilon \partial_x u_\varepsilon \partial_x^k u_\varepsilon + \Lambda_\varepsilon = 0, \tag{B.8}
$$

where

$$
\lambda_\varepsilon := k+1+(-1)^k(2k-1), \qquad \Lambda_\varepsilon := U_\varepsilon + \widetilde{P}_\varepsilon.
$$

Clearly for this equation we can apply the same argument used for the previous proofs.

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